DISS. ETH No. 14092, 2001

Ramsey-type results on planar geometric objects

A dissertation submitted to the
Swiss Federal Institute of Technology, ETH Zürich
for the degree of Doctor of Technical Sciences

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0.1 Abstract

In this thesis we focus on the problem of finding special subsets of sets of points and line segments in the plane.

A typical Ramsey-type result, in general, asserts the existence of a special substructure in a large structure. One of the first results in this field, for example, states that a 0-1 matrix of size $n \times n$ contains a submatrix of size $\log n \times \log n$ with only 0 or only 1 entries.

Usually the size of the special structure is much smaller than the whole structure. The Ackermann function occurs sometimes in those results. If the original set has some regularities, then a better, sometimes even linear, bound can be expected. Most of the results in this thesis are of this type, because of the structure of planar geometric objects. This is the case, for instance, for the intersection graph G_n of n line segments in the plane. While in a graph with n vertices, a complete or empty bipartite subgraph of size $\log n$ can be asserted in general, we will see that G_n always contains a complete or empty bipartite subgraph with size $n/330$.

We can not always use geometric arguments to find regular subsets, sometimes all what we know is that the intersection graph does not contain a specific graph as induced subgraph. This already implies that the intersection graph contains a relatively large homogeneous (complete or empty) subgraph. Here we can use tools from graph theory. In this thesis, beyond the geometric results, we will see an overview of results around the Erdős-Hajnal conjecture, which concerns the size of the largest homogenius subgraph.

0.2 Zusammenfassung

Die vorliegende Doktorarbeit befasst sich mit der Existenz spezieller Teilmengen von Punktmengen und Mengen von Linienstrecken der Ebene.

Typische Resultate der Ramsey-Theorie sind derart, dass sie die Existenz einer besonderen Teilstruktur in einer grossen Struktur bestätigen. So war eines der ersten Ergebnisse dieser Theorie die Erkenntnis, dass jede $n \times n$-dimensionale 0-1 Matrix eine $\log n \times \log n$ Untermatrix enthält, welche nur aus Nullen oder nur aus Einsen besteht.

Normalerweise ist die besondere Struktur sehr viel kleiner als die gesamte Menge. Die Ackermannfunktion taucht manchmal in diesen Resultaten auf. Falls für die Ausgangsmenge Irregularitäten bekannt sind, kann eine bessere, manchmal sogar lineare Schranke erwartet werden. Die meisten Resultate in dieser Arbeit sind dieser Art, dank der besonderen Struktur geometrischer Objekte in der Ebene. Beispielsweise können wir für den Schnittgraph G_n von n Strecken in der Ebene die Existenz eines vollständigen oder leeren bipartiten Teilgraphen der Grösse $n/330$ beweisen, dagegen enthält ein Graph mit n Eckpunkten nachweisbar nur einen vollständigen oder leeren bipartiten Teilgraphen der Grösse $\log n$.

Nicht immer ist ein geometrischer Ansatz sinnvoll, wenn wir reguläre Teilmengen finden wollen. Manchmal können wir nur die Nichtexistenz eines speziellen Graphen als induzierten Teilgraph des Schnittgraphen zeigen. Dies impliziert jedoch bereits, dass der Schnittgraph einen relativ grossen homogenen (vollständigen oder leeren) Teilgraphen enthält. An dieser Stelle können wir graphentheoretische Ansätze verwenden. Zusätzlich zu den geometrischen Resultaten geben wir in dieser Arbeit einen Überblick über Resultate der Erdős-Hajnal Behauptung, die sich auf die Grösse der grössten homogenen Subgraphen bezieht.

Contents

Introduction

Over the years the disciplines of computational and combinatorial geometry have been separated, but they are nevertheless closely related. There are results in combinatorial geometry where the main tools of the proofs stem from computational methods. On the other hand one can find many efficient algorithms in computational geometry where the key of the process is an observation or theorem from combinatorial geometry.

In this thesis we show results from both sides with emphasis on the combinatorial questions.

This thesis is organized as follows. In the first chapter I introduce a counting method frequently used in my research. The further chapters and sections are concerned with specific problems in discrete mathematics. Sometimes they are only weakly related to each other but they are congenial either in the topics or in the proof technique

In the second chapter we give results related to planar point sets. We start with the Erdős–Szekeres Theorem and continue with the famous problem of Erdős on distinct distances. Together with Csaba Tóth we substantially improved the best known lower bound on the number of distinct distances determined by n points in the plane.

The AMS Mathematics Subject Classification includes a special topic: 52C10 Erdős problems and related topics of discrete geometry.

Most questions studied in my thesis were raised by Erdős or were inspired by his theorems. This is partly explained by the fact that a large part of my thesis is based on joint research with János Pach, a well known disciple and collaborator of Paul Erdős.

The third chapter deals with geometric graphs, with graphs whose vertices are points in the plane and whose edges are line segments connecting these vertices. Geometric graph theory is a fast developing new area of research.

Many important problems in combinatorial geometry can be reformulated as problems for geometric graphs, and this setting often suggests new, fruitful approaches.

In the fourth chapter we analyze the intersection patterns of line segments. The problem of detecting and reporting all intersections in a collection of line segments in the plane is a fundamental problem in computational geometry. We establish some new Ramsey- and Szemerédi-type structure theorems for systems of segments, suggesting a new and perhaps efficient approach to this problem.

Finally we introduce a tool from graph theory with many interesting geometric applications.

Acknowledgments

First of all I want to thank my advisor Emo Welzl for offering me the great opportunity to complete my Ph.D. in his research group.

I would like to thank my previous advisor Miki Simonovits for his help to become a scientist.

Favored by fortune I have a third, informal advisor, János Pach. It is his achievement (or fault) that I re-started my research in discrete mathematics. I want to thank him for his help and I hope that we will have many other joint results in the future.

I am very grateful to Günter Rote for agreeing to be a co-examiner.

Let me express my thanks to friends and colleagues in our research group "gremo". Special thanks to Falk Tschirschnitz for proofreading this thesis. His suggestions helped to improve this work.

I am also grateful to Tamás Várady for the possibility to work in his research group at SZTAKI in Budapest.

I am glad to be the first to complete a Ph.D. in the joint Berlin/Zürich graduate program Combinatorics, Geometry, and Computation, financed by the German Science Foundation (DFG) and ETH Zürich.

A simple counting method

In this first section we introduce a simple counting method which is not new, but it turned out that it is remarkably useful when one is working with geometric objects. If the number of k-element subsets is large (k is fixed) in a set, then it contains sub-systems with special features.

In the following sections we will use it quite frequently, so here we outline the method with three simple examples, two from geometry and one from graph theory.

1.1 Monotone increasing point sets

Let C be an n-element set of points in the plane. Assume that no two points have the same x-coordinate or the same y-coordinate. A k-tuple $(c_1, c_2, \ldots, c_k)$, where $c_i = (x_i, y_i),\ i = 1, 2, \ldots, k$, is said to be *monotone increasing* (or *monotone decreasing*) if $x_1 < x_2 < \ldots < x_k$ and $y_1 < y_2 < \ldots < y_k$ (or $y_1 > y_2 > \ldots > y_k$, respectively).

A k-tuple of point sets $(C_1, C_2, \ldots, C_k)$ is said to be *monotone* if any $(c_1, c_2, \ldots, c_k)$ with $c_i \in C_i,\ i = 1, 2, \ldots, k$, is monotone increasing or monotone decreasing.

Theorem 1.1 *For every natural number k, there is a positive constant $\delta = \delta(k)$, such that for large enough n every n element planar point set C there is a k-tuple $(C_1, C_2, \ldots, C_k)$, of subsets of C, that is monotone and $|C_1| = |C_2| = \ldots = |C_k| = \lfloor \delta n \rfloor$.*

PROOF. For simplicity we will prove the theorem for $k = 3$. It is not difficult to see that the very same way works for any $k \in N$.

We need the following well-known observation of Erdős and Szekeres.

Lemma 1.1 *[35] Every sequence of $p^2 + 1$ reals contains either a monotone increasing or a monotone decreasing subsequence of length $p + 1$.*

The pair (c_2, c_4) is called the *support* of a 5-tuple $(c_1, c_2, c_3, c_4, c_5)$.

It follows from the lemma that among any 17 points in C there is a monotone increasing or a monotone decreasing 5-tuple. Thus, the total number of monotone increasing or monotone decreasing 5-tuples of C is at least

$$\frac{\binom{|C|}{17}}{\binom{|C|-5}{12}} = \frac{\binom{|C|}{5}}{\binom{17}{5}}.$$

Consequently, there are at least

$$\frac{\binom{|C|}{5}}{\binom{|C|}{2}\binom{17}{5}} = \frac{(C-2)(C-3)(C-4)}{371280} > \frac{1}{4 \times 10^5}|C|^3 \text{ (if } C \text{ is large enough)}$$

monotone increasing or decreasing 5-tuples that share the same support (c', c''). Suppose without loss of generality that the y-coordinate of c'' is larger than y', that is, all of these 5-tuples are monotone increasing. We define now the three sets as

$$C_1 = \{c_1 | \text{ there are } c_3, c_5 \in C : (c_1, c', c_3, c'', c_5) \text{ is monotone increasing}\}$$

$$C_2 = \{c_3 | \text{ there are } c_1, c_5 \in C : (c_1, c', c_3, c'', c_5) \text{ is monotone increasing}\}$$

$$C_3 = \{c_5 | \text{ there are } c_1, c_3 \in C : (c_1, c', c_3, c'', c_5) \text{ is monotone increasing}\}$$

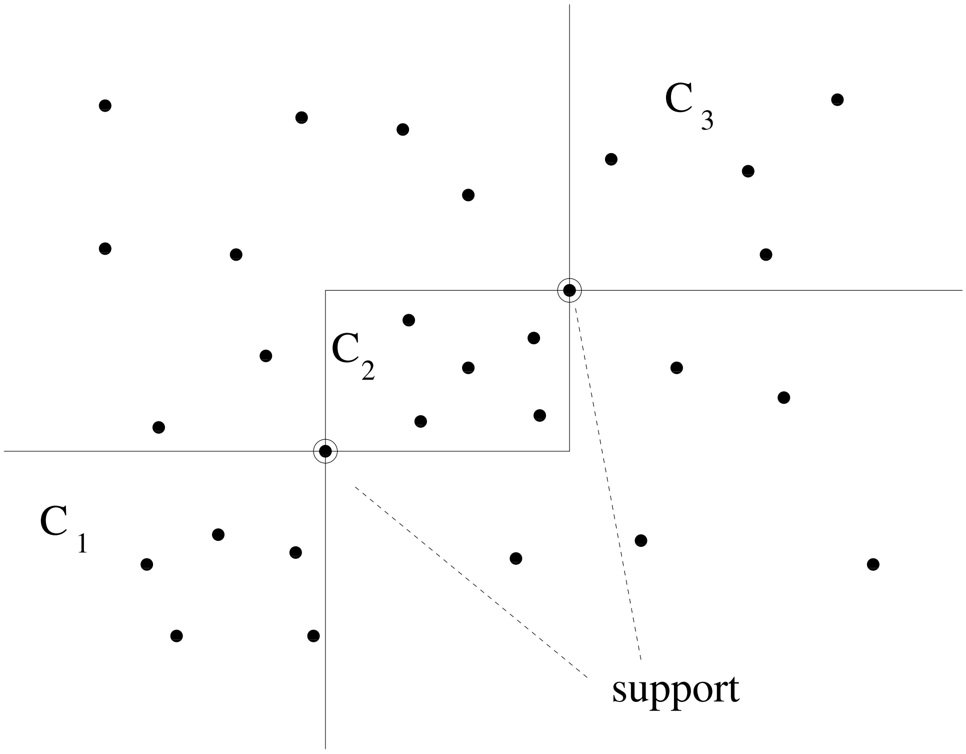

Figure 1.1

Those sets are disjoint by definition and

$$\frac{1}{4 \times 10^5}|C|^3 < |C_1||C_2||C_3| \leq \min(|C_1|, |C_2|, |C_3|)|C|^2$$

therefore

$$\min(|C_1|, |C_2|, |C_3|) > \frac{1}{4 \times 10^5}|C|$$

so

$$\delta(3) > \frac{1}{4 \times 10^5}\,.$$

1.2 Points in convex position

In their classical paper written in 1935, Erdős and Szekeres [34, 28] proved that for every $r \geq 3$, there exists an integer $f(r)$ such that any set of at least $f(r)$ points in the plane has r elements in convex position. This result has inspired extensive research in combinatorial and computational geometry (see e.g. [11, 43, 46, 69, 90]).

The value of $f(r)$ for $r > 6$ is unknown, but we have the following estimates

$$2^{r-2} + 1 \leq f(r) \leq \binom{2r-5}{r-2} + 2 < 4^r.$$

The lower bound is from the original Erdős Szekeres paper [34]. They conjectured that this is the exact value of $f(r)$. The upper bound was recently found by Tóth and Valtr [89].

It follows that if n is much larger than $f(r)$, then every n-element point set P contains many r-tuples in convex position. For instance, it was proved [81] that for a suitable constant $c_r > 0$, one can select a sequence of $c_r n$ distinct elements from P, such that any r consecutive members are in convex position. In the case $r = 4$, Nielsen [66] and, for general Bárány and Valtr [8] proved the following stronger result.

Theorem 1.2 *[8] For any fixed $r \geq 4$, there is a constant $c_r > 2^{-2^{6r}}$ satisfying the following condition.*

Every n-element point set P in general position, n large enough, in the plane has r pairwise disjoint subsets P_i $(1 \leq i \leq r)$ such that $|P_i| \geq \lfloor c_r n \rfloor$ and no matter how we pick one point from each P_i we obtain a set in convex position.

This result provides a *canonical* way to find many convex r-gons in a sufficiently large point set in the plane.

Here we will see that one can prove this theorem by our counting argument in a simple way, and with a better constant, 2^{-8r^2} instead of $2^{-2^{6r}}$.

PROOF. Let us label the vertices of a convex $2r$-gon C_{2r} by $1, 2, \ldots, 2r$ in clockwise order. The set of points having odd labels and the set of points with even labels are called *holders of* C_{2r}. Clearly every convex $2r$-gon has exactly two holders and each of them has r elements.

The number of convex $2r$-gons in an n element set is at least

$$\binom{n}{f(2r)}\binom{n-2r}{f(2r)-2r}^{-1} = \binom{n}{2r}\binom{f(2r)}{2r}^{-1}$$

and the number of holders is at most $\binom{n}{r}$. So there is an r-tuple R that it is holder of at least

$$2\binom{n}{2r}\binom{f(2r)}{2r}^{-1}\binom{n}{r}^{-1}$$

different convex $2r$-gons in our set. This number is larger than $f(2r)^{-2r}n^r$ if n is sufficiently large compared to r.

Now we label the vertices of the convex $2r$-gons which have the holder R. Let us start with a fixed point from R, it has label 1. In each convex $2r$-gon label the remaining vertices in clockwise order by $2, 3, \ldots, 2r$.

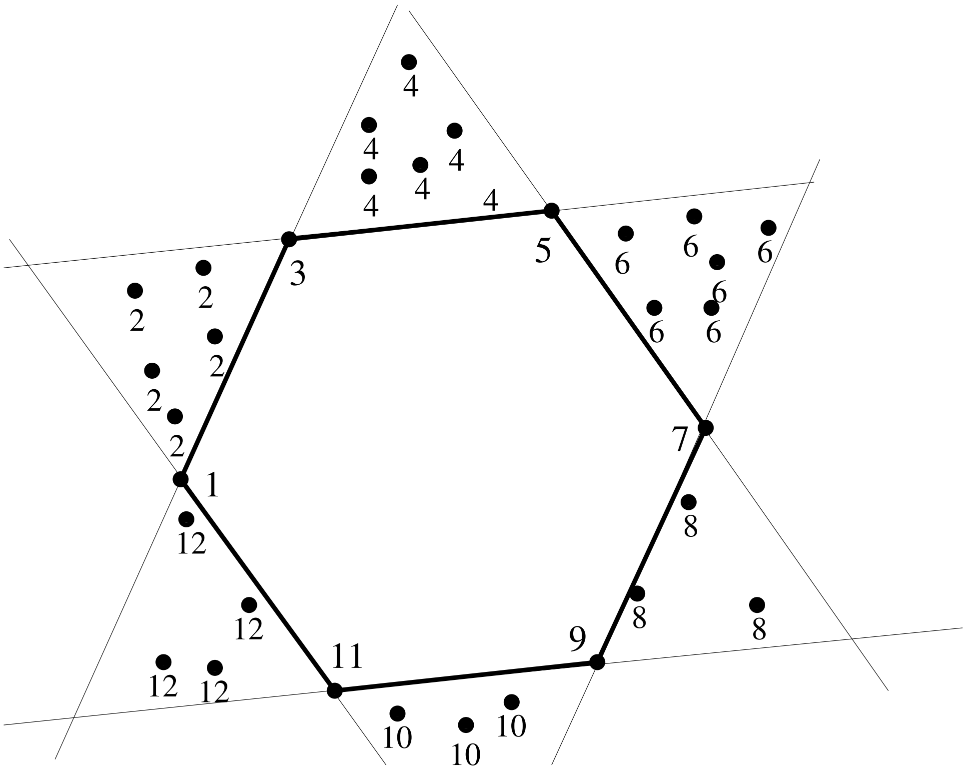

Figure 1.2 Labeling in case $2r = 12$

Let us define P_i as $P_i = \{p \in P : p \text{ has label } 2i\}$, $(1 \leq i \leq r)$. These sets are disjoint sets; therefore

$$f(2r)^{-2r}n^r < \prod_{i=1}^{r} |P_i| < n^{r-1}(\min_{1\leq i\leq r} |P_i|)$$

and we have

$$\min_{1\leq i\leq r} |P_i| > f(2r)^{-2r}n > 2^{-8r^2}n.$$

To complete the proof, all what we need is a simple observation: no matter how we choose one point in each P_i $(1 \leq i \leq r)$, the points are in convex position.

1.3 Fat trees in dense graphs

Here we show that a modified version of the above counting method can be useful in other fields of combinatorics.

Let G be a graph. For any positive integer r, let $G(r)$ denote the graph obtained from G by replacing each vertex v of G by r vertices, v_i, $(1 \leq i \leq r)$, and connecting v_i and u_j by an edge if and only if vu is an edge in G, $(1 \leq i, j \leq r)$. In this way we blow up the graph. If the graph is a tree then the resulting graph is a “fat tree”. The next theorem shows that every dense graph contains many fat trees.

Theorem 1.3 *For every $c > 0, r \geq 1$, there exists a constant $c_{r,c} > 0$ with the following property.*

Let T be any tree of at most $c_{r,c}n$ vertices. Then every graph G with n vertices and at least cn^2 edges has a subgraph isomorphic to $T(r)$.

PROOF. Let G be a graph, whose vertex set and edge set are denoted by $V(G)$ and $E(G)$, respectively. Assume that $|V(G)| = n$ and $|E(G)| \geq cn^2$ for some constant $c > 0$, and let r be a fixed positive integer.

First, we show that G contains many complete bipartite subgraphs $K_{r,r}$ with r. The proof is based on the following statement, discovered by Erdős, which is a weak version of a result of Kővári, Sós and Turán [50].

Lemma 1.2 *For every $r \geq 1$ and every $\gamma > 0$, there exists a positive integer $n_0 = n_0(r, \gamma)$ with the following property.*

Every graph with n_0 vertices and at least γn_0^2 edges contains a $K_{r,r}$ as subgraph.

Let x denote the number of n_0-element subsets of $V(G)$ which induce a subgraph of G with at least γn_0^2 edges. Counting the edges with multiplicity we have

$$x\binom{n_0}{2} + \left(\binom{n}{n_0} - x\right)\gamma n_0^2 > |E(G)|\binom{n-2}{n_0-2} \geq cn^2\binom{n-2}{n_0-2},$$

which yields

$$x > \binom{n}{n_0}\frac{c(n_0-1)-\gamma n_0}{\frac{n_0-1}{2}-\gamma n_0}.$$

Thus, for $\gamma = c/2, n_0 > 2$, we obtain $x > \frac{c}{2}\binom{n}{n_0}$.
Set $n_0 := n_0(r,\gamma) = n_0(r, c/2)$. By the lemma, every subgraph of G with n_0 vertices and at least $(c/2)n_0^2$ edges contains at least one copy of $K_{r,r}$. Thus, the number y of complete bipartite subgraphs $K_{r,r}$ of G satisfies

$$y \geq \frac{x}{\binom{n-2r}{n_0-2r}} > \frac{\frac{c}{2}\binom{n}{n_0}}{\binom{n-2r}{n_0-2r}} > \frac{cn^{2r}}{2n_0^{2r}}.$$

Suppose for simplicity that n is divisible by r, and consider all possible partitions of $V(G)$ into classes of size r. The number of these partitions is

$$p(n,r) = \frac{\binom{n}{r}\binom{n-r}{r}\binom{n-2r}{r}\cdots\binom{r}{r}}{(n/r)!}.$$

For every partition P, construct a graph $G(P)$ whose vertices are the classes V_i $(1 \leq i \leq n/r)$ of the partition, and two vertices V_i and V_j are connected by an edge of $G(P)$ if and only if G contains all edges running between them. By averaging over all partitions, we find that there exists a P such that the number of edges of $G(P)$ is at least

$$\frac{yp(n-2r,r)}{p(n,r)} > \frac{cn^{2r}}{2n_0^{2r}}\frac{\binom{\frac{n}{r}}{r}\binom{\frac{n-r}{r}}{r}}{\binom{n}{r}\binom{n-r}{r}} > \frac{cn^{2r}}{2n_0^{2r}}\frac{\binom{\frac{n}{r}}{r}\binom{\frac{n-r}{r}}{r}}{\left(\frac{en}{r}\right)^r\left(\frac{e(n-r)}{r}\right)^r}$$

which is at least

$$\frac{c}{2r^2}\left(\frac{r}{en_0}\right)^{2r}\cdot n^2.$$

We can now finish the proof of the theorem by applying to $G(P)$ the following assertion.

Lemma 1.3 *For any $C > 0$, every graph with N vertices and at least CN^2 edges contains every tree of at most CN vertices as a subgraph.*

PROOF. First we prove that the graph G contains a subgraph with minimal degree at least CN. Let us remove the vertex with minimal degree from G if it is smaller than CN. From the new graph let us remove a vertex with minimal degree if it is smaller than CN and continue this procedure until every vertex in the remaining graph G' has degree at least CN. (We can not remove every vertex in this way because in every step we remove less than CN edges, but G has CN^2 edges.)

Now if we have a tree of size CN then we can simply embed it to G' by starting with an arbitrary vertex, continuing with the neighbors of the vertex, etc.

Hence, the theorem is true with

$$c_{r,c} = \frac{c}{2r^2}\left(\frac{r}{en_0}\right)^{2r}$$

The existence part of this theorem can be derived from Szemerédi's Regularity Lemma [85] too, but, counting the value of n_0 (see in [50]), one can see that the constant would be much worse and this proof is relatively simple.

Point sets

2.1 Erdős-Szekeres Theorem

In this section we will see Erdős-Szekeres type results for convex sets.

Bisztriczky and Fejes Tóth [12] found the following generalization of the Erdős-Szekeres theorem to families of pairwise disjoint compact convex sets in the plane. We say that such a family $\mathcal{F}$ is in *general position* if none of its members is contained in the convex hull of the union of two others. We say that $\mathcal{F}$ is in *convex position* if none of its members is contained in the convex hull of the union of the others.

Theorem 2.1 *[12] For every $r \geq 3$, there exists an integer $g(r)$ such that any family of at least $g(r)$ pairwise disjoint compact convex sets in general position in the plane has r members in convex position.*

In this section, we apply the counting argument introduced in the first chapter to establish the following common generalization of the Theorem 1.2 and Theorem 2.1.

Theorem 2.2 *For every $r \geq 4$, there is a positive constant $c_r = 2^{-O(r^2)}$ with the following property.*

Every family $\mathcal{F}$ of n pairwise disjoint compact convex sets in general position in the plane has r disjoint $\lfloor c_r n \rfloor$-element sub-families $\mathcal{F}_i$ $(1 \leq i \leq r)$ such that no matter how we pick one set from each $\mathcal{F}_i$, they are always in convex position.

PROOF. The proof is based on a stronger version of Theorem 2.1 .

Lemma 2.1 *[73] For every $k \geq 3$, any family of 2^{4k} pairwise disjoint compact convex sets in general position in the plane has k members in convex position.*

Let $\mathcal{F}$ be a family of n compact convex sets in the plane. Assume without loss of generality that no two members of $\mathcal{F}$ have a common vertical tangent line. For $C \in \mathcal{F}$, let $\pi(C)$ denote the projection of C onto the x-axis. Following [55], we define four partial orders, $\prec_1$, $\prec_2$, $\prec_3$ and $\prec_4$, on $\mathcal{F}$. For any two disjoint sets $A, B \in \mathcal{F}$,

1. $A \prec_1 B$ if $\pi(A) \subseteq \pi(B)$ and A lies below B ("below" w.r.t. direction of the y-axis).
2. $A \prec_2 B$ if $\pi(A) \subseteq \pi(B)$ and A lies above B.
3. $A \prec_3 B$ if the left endpoint of $\pi(B)$ is to the right of the left endpoint of $\pi(A)$, the right endpoint of $\pi(B)$ is to the right of the right endpoint of $\pi(A)$ and in the part where $\pi(A)$ and $\pi(B)$ overlap (if any), A lies above B.
4. $A \prec_4 B$ if the left endpoint of $\pi(B)$ is to the right of the left endpoint of $\pi(A)$, the right endpoint of $\pi(B)$ is to the right of the right endpoint of $\pi(A)$ and in the part where $\pi(A)$ and $\pi(B)$ overlap (if any), A lies below B.

Lemma 2.2 *[55]. Any family of more than $(k-1)^4(s-1)$ compact convex sets in the plane, no s of which have pairwise nonempty intersections, contains k members that form a chain with respect to one of the relations $\prec_1$, $\prec_2$, $\prec_3$, $\prec_4$.*

Applying first Lemma 2.2 and then Lemma 2.1, we obtain that for every k, any 2^{16k}-membered sub-family of $\mathcal{F}$ contains k sets in convex position that form a chain with respect to one of the relations $\prec_j$ $(1 \leq j \leq 4)$.

Set $k = 4r - 2, K = 2^{16k}$. Then it follows that there exists $1 \leq j \leq 4$ such that $\mathcal{F}$ has at least $\frac{1}{4}\binom{n}{k}/\binom{K}{k}$ chains $\mathcal{B} = (B_1 \prec_j B_2 \prec_j \ldots \prec_j B_k)$, whose members are in convex position. We distinguish two substantially different cases according to the value of j.

Case 1: $j = 1$. Let $\mathcal{B}$ be any chain of length $k = 4r - 2$ with respect to $\prec_1$, whose members are in convex position. Then $\mathcal{B}$ has a subchain $\mathcal{C} =$

$(C_1, C_2, \ldots, C_{2r-1})$ with the following property. Each C_i contributes to the boundary of the convex hull $\operatorname{conv} \cup_{i=1}^{2r-1} C_i$ at least one point to the left of C_1, or each C_i contributes to $\partial(\operatorname{conv} \cup_{i=1}^{2r-1} C_i)$ at least one point to the right of C_1. In the former case we call $\mathcal{C}$ a *left-convex* chain and in the latter one a *right-convex* chain. Thus, there are at least

$$\frac{\frac{1}{4}\binom{n}{k}/\binom{K}{k}}{2\binom{n-2r+1}{2r-1}}$$

different chains

$$\mathcal{C} = (C_1 \prec_1 C_2 \prec_1 \ldots \prec_1 C_{2r-1})$$

of the same type, say, left-convex. Define the *support* of $\mathcal{C}$ as the subchain $\mathcal{C}^* \subseteq \mathcal{C}$ formed by the even-numbered elements, i.e., let

$$\mathcal{C}^* = (C_2, C_4, C_6, \ldots, C_{2r-2}).$$

Clearly, there are at least

$$\frac{\frac{1}{4}\binom{n}{k}/\binom{K}{k}}{2\binom{n-2r+1}{2r-1}\binom{n}{r-1}} > \frac{n^r}{K^{4r-2}}$$

different left-convex chains $\mathcal{C} = (C_1, C_2, \ldots, C_{2r-1})$ which have the same support. These chains are called *standard*. Let

$$(\bar{C}_2, \bar{C}_4, \bar{C}_6, \ldots, \bar{C}_{2r-2})$$

denote the common support of the standard chains. We will refer to this sequence as the *standard support*.

For any t $(1 \leq t \leq r)$, let $\mathcal{F}_t$ denote the family of all members of $\mathcal{F}$ that occur as the $(2t-1)$-st element $C_{2t-1} \in \mathcal{C}$ for some standard chain $\mathcal{C}$. We have

$$|\mathcal{F}_t| > \frac{n^r}{K^{4r-2}} \prod_{s \neq t} |\mathcal{F}_s|^{-1} > \frac{n}{K^{4r-2}}.$$

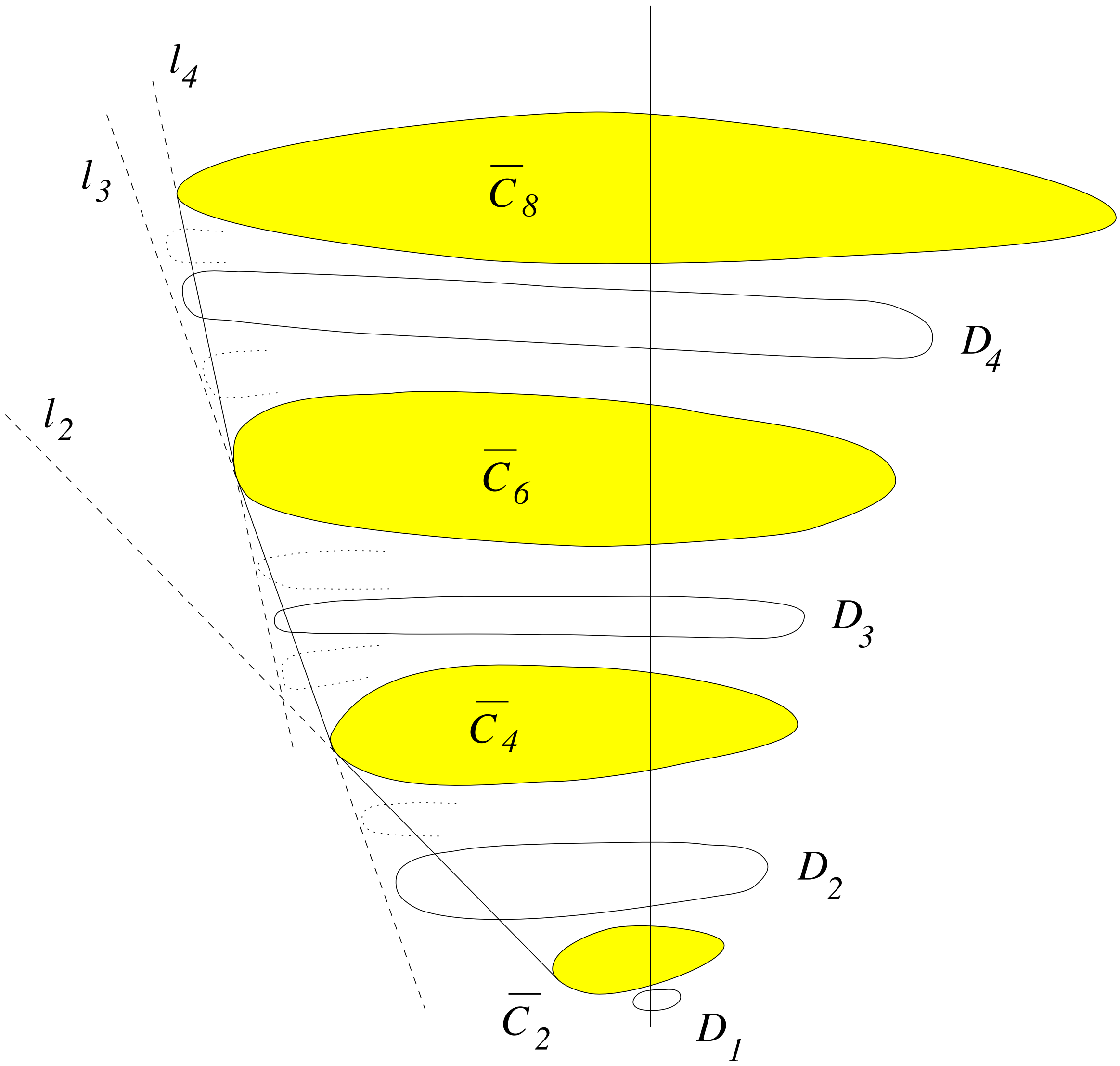

Figure 2.1 $D_1, \ldots, D_r$ **are in convex position**

It remains to show that for every choice $D_t \in \mathcal{F}_t$ $(1 \leq t \leq r)$, the sets $D_1, D_2, \ldots, D_r$ are in convex position (cf. Figure 2.1). To see this, consider the left-hand side ∂ of the boundary of the union of all members of the standard support. ∂ consists of nonempty portions of the boundaries of the sets $\bar{C}_{2t}$ $(1 \leq t < r)$, separated by straight-line segments. For any $1 < t < r$, let l_t denote the the common tangent line of the sets $\bar{C}_{2t-2}$ and $\bar{C}_{2t}$ with the property that every other member of the standard support is on its right-hand side. To finish the proof in Case 1, it is sufficient to notice that every $D_t \in \mathcal{F}_t$ has at least one point to the left of l_t, while all members of $\cup_{s \neq t} \mathcal{F}_s$ lie on the right-hand side of l_t. Therefore, $D_1, D_2, \ldots, D_r$ are in convex position.

Case 2: $j = 3$. Let $\mathcal{B}$ be any chain of length $k = 4r - 2$ with respect to $\prec_3$, whose members are in convex position. Then $\mathcal{B}$ has a subchain $\mathcal{C}$ =

$(C_1, C_2, \ldots, C_{2r-1})$ with the following property. Each C_i contributes at least one point to the upper portion of $\mathrm{bd\,conv} \cup_{i=1}^{2r-1} C_i$, or each C_i contributes at least one point to the lower portion of $\mathrm{bd\,conv} \cup_{i=1}^{2r-1} C_i$. In the former case, $\mathcal{C}$ is called an *upper-convex* chain, and in the latter one, a *lower-convex* chain. Thus, there are at least

$$\frac{\frac{1}{4}\binom{n}{k}/\binom{K}{k}}{2\binom{n-2r+1}{2r-1}}$$

different chains $\mathcal{C} = (C_1 \prec_3 C_2 \prec_3 \ldots \prec_3 C_{2r-1})$ of the same type, say, upper-convex. The rest of the argument is exactly the same as in Case 1, with the only difference that in place of the left-hand side ∂ of the boundary of the union of all members of the standard support, we have to consider its *upper side*.

The cases $j = 2$ and $j = 4$ are symmetric counterparts of the above two cases, so they do not have to be treated separately.

We close this section with a generalization of Theorem 2.1 .

We call a set S of convex sets in *concave position*, if the convex hull of any subset of S is a convex hull of at most two sets of S.

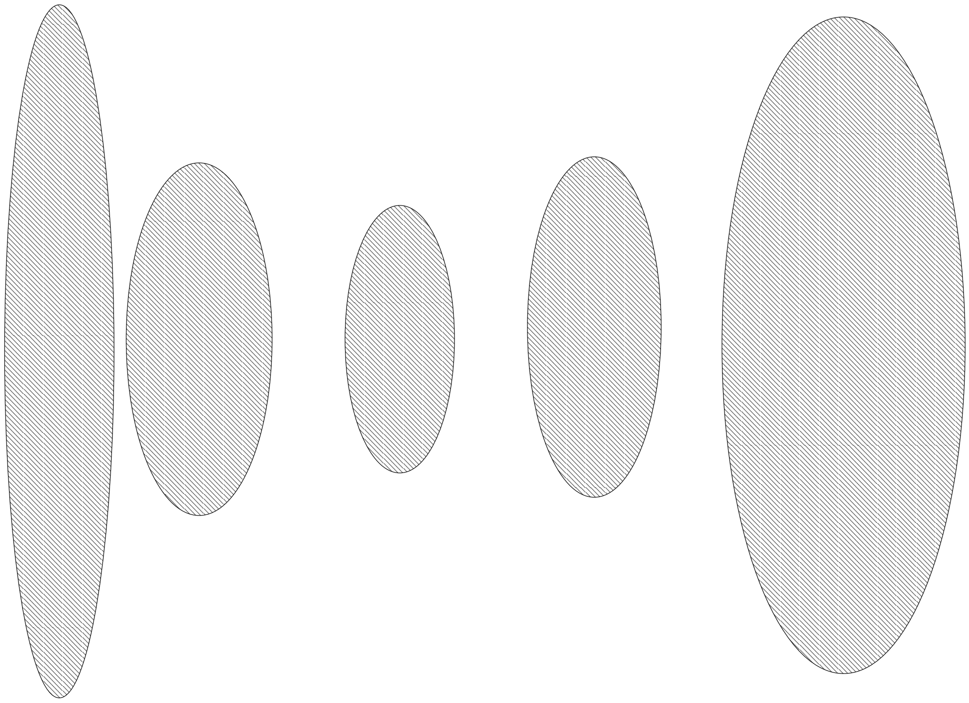

Figure 2.2 Sets in concave position

Theorem 2.3 *For every $r \geq 3$, there exists an integer $g(r)$ such that any* *family of at least $g(r)$ pairwise disjoint compact convex sets in the plane has r members in convex or in concave position.*

We remind the reader of the fact that while in the original Bisztriczky–Fejes Tóth Theorem there is the extra condition, that the sets are in "general" position, in this theorem their position is arbitrary.

PROOF. Let us color the triples of the sets by two colors, say blue and red, according to the position they are in, concave or convex. From Ramsey's Theorem one can find a large subset where all the triples are from the same color. This subset has to be convex or concave.

2.2 Almost straight lines

There is another Erdős-Szekeres theorem for point sets, "only" 40 years old, on so-called almost straight lines [35]. Using the counting method described in the first chapter, one can prove a fractional version of this theorem. A polygonal path $p_1 p_2 \dots p_r$ in d-space is called *ε-straight* if $\angle p_{i-1} p_i p_{i+1} > \pi - \varepsilon,\ 1 < i < r$ (cf. [35, 68]). We define the *length* of a polygonal path to be the number of its *vertices*.

Theorem 2.4 *For every $d \geq 2, r \geq 3$ and $\varepsilon > 0$, there exists a positive constant $c = c^d_{r,\varepsilon}$ with the following property.*

Every n-element point set P in general position in Euclidean d-space has r pairwise disjoint subsets P_i $(1 \leq i \leq r)$ with at least $\lfloor cn \rfloor$ elements each, such that no matter how we pick a point from each P_i, they always form an ε-straight polygonal path.

PROOF. To establish the theorem, we need the following generalization of a result from [35].

Lemma 2.3 *[68] There exists a constant $c > 0$ such that any set of at least $k^{(c/\varepsilon)^{d-1}}$ points in Euclidean d-space has k elements that form an ε-straight polygonal path of length k.*

Let ε, d, and r be fixed, and set $k = 2r - 1$. By Lemma 2.3, there exists an integer $K = K(\varepsilon, d, r)$ such that any set of K points in d-space contains

k elements that form the vertex set of an $\varepsilon/3$-straight polygonal path Π. If we skip every other vertex of P, then we obtain a polygonal path Π' with r vertices, which is ε-straight. The sequence formed by the $r-1$ vertices we skipped is called the *support* of Π.

Consider now any set P of n points in the plane. P contains at least

$$\binom{n}{K}\binom{n-k}{K-k}^{-1} = \binom{n}{k}\binom{K}{k}^{-1}$$

different $\varepsilon/3$-straight polygonal paths of length k, and at least

$$\frac{\binom{n}{k}/\binom{K}{k}}{n!/(n-r+1)!} > \frac{n^r}{K^{2r-1}}$$

of them must share the same support S.

Let P_i denote the set of all elements of P that occur as the $(2i-1)$-st vertex in some $\varepsilon/3$-straight polygonal path of length k, whose support is S $(1 \le i \le r)$. These sets meet the requirements of the theorem, and for every i, we have

$$|P_i| > \frac{n^r}{K^{2r-1}} \prod_{j \neq i} |P_j|^{-1} > \frac{n}{K^{2r-1}}.$$

This concludes the proof of Theorem 2.4.

The last theorem in this section deals with convex sets again.

Theorem 2.5 *For every $r, s \ge 2$, there exists a positive constant $c = c_{r,s} = (rs)^{-O(r)}$ with the following property.*

Let $\mathcal{F}$ be a family of n compact convex sets in the plane, no s of which have pairwise nonempty intersections. Then $\mathcal{F}$ has r disjoint $\lfloor cn \rfloor$-membered subfamilies $\mathcal{F}_i$ $(1 \le i \le r)$ such that no two sets belonging to distinct subfamilies have a point in common.

PROOF. Setting $K = (k-1)^4(s-1)+1$ and $k = 2r-1$, we obtain, just like in the proof of Theorem 2.2, that there exists $1 \le j \le 4$ such that $\mathcal{F}$ has at least $\frac{1}{4}\binom{n}{k}/\binom{K}{k}$ chains $\mathcal{C}$ of length k with respect to $\prec_j$. If we skip every other element of $\mathcal{C}$, we obtain a chain $\mathcal{C}'$ of length r. The chain $\mathcal{C} \setminus \mathcal{C}'$ is called the *support* of $\mathcal{C}$. It follows that at least

$$\frac{\frac{1}{4}\binom{n}{k}/\binom{K}{k}}{\binom{n}{r-1}} > \frac{n^r}{K^{2r-1}}$$

chains C share the same support S.

For every i $(1 \leq i \leq r)$, let $\mathcal{F}_i$ denote the set of all members of $\mathcal{F}$ that occur as the $(2i-1)$-st smallest element of a chain in $(\mathcal{F}, \prec_j)$ with length k and support S. It is clear that no two sets belonging to distinct $\mathcal{F}_i$'s have a point in common. The same estimation as at the end of the proof of Theorem 2.2 gives that $|\mathcal{F}_i| \geq \frac{n}{K^{2r-1}}$ for every i.

It is possible that the following far-reaching generalization of Theorem 2.5 is also true. For every $s \geq 2$, there exists a constant $c = c_s > 0$ with the property that any family of n compact connected sets in the plane, no s of which have pairwise nonempty intersections, has at least cn pairwise disjoint members.

2.3 Different distances between n points

The following well known problem is due to Paul Erdős [26] (see also [69]): Given n distinct points in the plane, what is the minimum number of distinct distances determined by them?

Denoting this minimum by $G(n)$, Erdős [26] conjectured that

$$G(n) = \Omega(n/\sqrt{\log n})$$

and showed that this bound is attained by the $\sqrt{n} \times \sqrt{n}$ grid. Moser [63] proved that the number of distinct distances is at least $\Omega(n^{2/3})$. Chung [17] and Chung, Szemerédi, and Trotter [18] improved this bound to $\Omega(n^{5/7})$ and $\Omega(n^{4/5}/\log^c n)$, respectively, where c is a small positive constant. Finally, Székely [83] proved that $G(n) = \Omega(n^{4/5})$, and that there always exists at least one point from which there are at least $\Omega(n^{4/5})$ distinct distances. Our main result is

Theorem 2.6 *Any set P of n points in the plane has an element from which the number of distinct distances to the other points of P is at least $\Omega(n^{6/7})$.*

The proof relies on three results: (a) Beck's theorem [9] on the minimum number of lines connecting points in a planar point set; (b) the Szemerédi-Trotter theorem [84] on the number of incidences between points and lines; and (c) Székely's method [83] for estimating the number on incidences using a previously known upper bound on crossing numbers of graphs drawn in the plane. It is worth noting that these three ingredients are related to each other. Beck's result follows from the Szemerédi-Trotter theorem, which, in turn, has a very elegant proof using Székely's method.

A topological (multi-)graph is a (multi-)graph $G(V, E)$ drawn in the plane such that the vertices of G are represented by distinct points in the plane, and its edges by continuous arcs between the corresponding point pairs. Any two arcs representing distinct edges have finitely many points in common. We will make no notational distinction between the vertices (resp., edges) and the points (resp., arcs) representing them.

Unlike in the standard definition of topological graphs, we allow arcs representing edges of G to pass through other vertices. Such topological graphs were first employed by Pach and Sharir [71]. Two edges of a topological graph are said to form a *crossing*, if they have a common point which is not an endpoint of both curves. Thus, two crossing edges together have four distinct endpoints. The *crossing number* of a *topological* graph or multigraph is the total number of crossing pairs of edges. The *crossing number* of an *abstract* graph or multigraph G is the minimum crossing number over all possible representations (i.e., drawings) of G as a topological graph.

Proof of Theorem 2.6

Let P be a set of n points in general position in the plane. Let t be the maximal number of distinct distances measured from one point, that is $t = \max_{p\in P} |\{|pq| \;:\; q \in P\}|$. Suppose that $t = o(n)$ and $t \geq c_1 n^{3/4}$, for some constant $c_1 > 0$. This latter assumption follows from the earlier results [18, 20]. We apply Beck's theorem to the point set P.

Theorem 2.7 (Beck [9]) *Given n points in the plane, at least one of the following two statements holds:*

1. *There is a line incident to at least $n/100$ points.*
2. *There are at least $\Omega(n^2)$ lines incident to at least two points.*

Let Ł denote the set of all lines connecting at least two points of P. We have

Corollary 2.8 *There is an absolute constant $c_0 > 0$ with the property that the number of points in P incident to at least $c_0 n$ distinct lines of Ł is at least $c_0 n$, provided that $t < n/100$.*

Let B denote the set of points in P incident to at least $c_0 n$ lines of Ł. By Corollary 2.8, $|B| \geq c_0 n$.

Fix a point $a \in B$. Let $P_a \subset P \setminus \{a\}$ be a maximal set such that for each point $q \in P_a$ the line aq contains no other point of P_a. Consider the set $\mathbb{C}_a$

of all circles centered at $a \in B$ that contain at least 3 points of P_a. Let P'_a denote the set of all elements of P_a which belong to a circle in $\mathbb{C}_a$. Clearly, $|P'_a| = \Omega(n)$, because $|P_a| = \Omega(n)$, and $t = o(n)$, by assumption.

After deleting at most two points from each circle in $\mathbb{C}_a$, partition the remaining points into pairwise disjoint consecutive triples (q_1, q_2, q_3). Clearly, the number of such triples T_a over all circles around a is $\Omega(n)$.

A line ℓ is called *rich* if ℓ is incident to at least m points in P, where m is a number to be specified later. A triple (q_1, q_2, q_3) is said to be *good* if the bisector of at least one of the segments q_1q_2, q_1q_3, or q_2q_3 is not rich; otherwise it is called *bad*. A point $p \in B$ is *good*, if at least half of the triples T_p are good, otherwise it is *bad*. Denote by g the number of good points in B.

The outline of the rest of the proof is as follows. First, we choose an m so that at least half of the points in B will be bad. Then we deduce a lower bound on the number of rich lines incident to a fixed bad point, which then implies a *lower* bound for the overall number of incidences between bad points and rich lines. On the other hand, a theorem of Szemerédi and Trotter, mentioned in the introduction, provides an *upper* bound on the same quantity. Comparing the two bounds, the desired inequality for t will follow.

Define a topological multigraph G on the vertex set $V = P$, as follows. If a triple (q_1, q_2, q_3) is good, add to the graph one edge between a pair of points from $\{q_1, q_2, q_3\}$ whose bisector is not rich. We generate exactly one edge for each good triple. Draw each such edge along the circle determined by the triple.

The number of vertices of G is $|V| = n$; the number of edges of G is $E = g \cdot \Omega(n) = \Omega(gn)$. The graph G may have multiple edges when two points, u and v, happen to belong to more than one good triple, associated with different points of B (as centers of the corresponding circles). However, the multiplicity of each edge is at most m, because all of these points of B must lie on the perpendicular bisector of u and v, which, by assumption, is not rich.

The following lemma of [83] is a straightforward extension of a result of Ajtai, Chvátal, Newborn, and Szemerédi [1] and of Leighton [58], to topological *multi*graphs. As we pointed out in the introduction, we use a slightly non-standard definition of topological multigraphs, which allows edges to pass through vertices, but Székely's proof applies *verbatim* to this case as well.

Lemma 2.4 (Székely [83]) *Let $G(V, E)$ be a topological multigraph, in which every pair of vertices is connected by at most m edges. If $|E| \geq 5|V|m$, then* *the crossing number of G is*

$$cr(G) \geq \frac{\beta|E|^3}{m|V|^2},$$

for an absolute constant $\beta > 0$.

Apply Lemma 2.4 to the graph G defined above, with

$$m = c_2 n^2/t^2,$$

where $c_2 > 0$ is a small constant. We distinguish two cases. If the condition in the lemma is not satisfied, then

$$\Omega(gn) = |E| < 5|V|m = 5c_2 n^3/t^2 \leq (5c_2/c_1^2)n^{3/2},$$

and, by choosing c_2 sufficiently small, we have $g \leq (c_0/2)n$. Otherwise, according to the statement,

$$cr(G) \geq \frac{\beta g^3 n^3}{(c_2 n^2/t^2) \cdot n^2} = \frac{\beta g^3 t^2}{c_2 n}.$$

On the other hand, since the edges of G are constructed along at most nt circles (at most t concentric circles around each point), and two circles have at most two common points, we clearly have

$$cr(G) \leq 2 \cdot \binom{nt}{2} \leq n^2 t^2.$$

Comparing the last two inequalities, we obtain, just as in the previous case, that $g \leq (c_0/2)n$, provided that c_2 is chosen sufficiently small.

Therefore, we can conclude that B has at least $\Omega(n)$ bad points.

Next, we estimate the number of rich lines incident to each bad point. For this, we need the following simple lemma.

Lemma 2.5 *Let T be a set of N triples (a_i, b_i, c_i) of $3N$ distinct real numbers such that $a_i < b_i < c_i$ for $i = 1, \ldots, N$, and assume that $c_i < a_{i+1}$ for all but at most $t - 1$ indices i. Let $W = \{(a_i + b_i)/2, (a_i + c_i)/2, (b_i + c_i)/2 \mid i = 1, \ldots, N\}$. Then*

$$|W| = \Omega\left(\frac{N}{t^{2/3}}\right).$$

This bound cannot be improved.

PROOF. Let the *range* of a triple $(a, b, c) \in T$ be defined as the interval $[a, c]$. By assumption, the sequence $(a_1, b_1, c_1, a_2, b_2, c_2, \ldots, a_N, b_N, c_N)$ can be partitioned into at most t contiguous monotone increasing subsequences. Partition the real axis into $N/(2t)$ open intervals so that each interval fully contains the ranges of t triples. (These intervals are constructed from left to right. Let x denote the right endpoint of the rightmost interval constructed so far. Discard the at most t triples whose ranges contain x, and move to the right until we reach a point y that lies to the right of exactly t new ranges. We add (x, y) as a new open interval, and continue in this manner until all triples are processed.)

Let s be one of these open intervals. Each triple in T, whose range is fully contained in s, contributes three elements to $W \cap s$, and no two triples of T contribute the same triple to $W \cap s$, as can be easily verified. It follows that $|W \cap s| \geq t^{1/3}$, or else the number of distinct triples of its elements will be smaller than t. Since the number of intervals s is $N/(2t)$, the inequality in the lemma follows. An easy construction shows that this result is tight: Let us choose algebraically independent numbers for W and generate a triple of N from every triple.

Apply Lemma 2.5 to the system of $\Omega(n)$ disjoint bad triples along the circles centered at a fixed bad point $a \in B$. Each point u which participates in such a bad triple is mapped to the *orientation* of the ray $\vec{au}$, i.e., to the counterclockwise angle between the positive x-axis and $\vec{au}$. By the construction of P_a, this mapping is an injection of P_a into $\mathbb{R}$. There are at most t bad triples whose ranges are mapped into an interval containing 0, and we discard all of them. The images of the remaining bad triples form a set of $N = \Omega(n)$ triples meeting the requirements of Lemma 2.5. Notice that there are at most two orientations in W that correspond to the same rich line trough a. Hence, Lemma 2.5 implies that, for each bad point a, the number of rich lines incident to a is at least $\Omega(n/t^{2/3})$.

Therefore, the number I of incidences between bad points and rich lines, satisfies

$$I = \Omega(n^2/t^{2/3}). \tag{2.1}$$

The same number can be estimated from above, using the following theorem of Szemerédi and Trotter, which comes in two equivalent formulations, both stated below.

Theorem 2.9 (Szemerédi-Trotter [84]) *(a) Given n distinct points in the plane, the number L_m of lines incident to at least $m > 2$ points is*

$$L_m = O\left(\frac{n^2}{m^3} + \frac{n}{m}\right).$$

(b) Given n distinct points and ℓ distinct lines in the plane, the number of point-line incidences is

$$I(n, \ell) = O(n^{2/3}\ell^{2/3} + n + \ell).$$

Both of these bounds are asymptotically tight.

Since $t \geq c_1 n^{3/4}$, we have $m = c_2 n^2/t^2 \leq (c_2/c_1^2)n^{1/2} = O(n^{1/2})$. Thus, by Theorem 2.9(a), the number L_m of rich lines satisfies $L_m = O(n^2/m^3) = O(t^6/n^4)$. This, in turn, implies, by part (b) of the same theorem, that the number I of incidences between bad points and rich lines, satisfies

$$I = O(n^{2/3}L_m^{2/3} + n + L_m) = O(t^4/n^2 + t^6/n^4 + n) = O(t^4/n^2). \quad (2.2)$$

Comparing (2.1) and (2.2), we obtain that $t = \Omega(n^{6/7})$, as required. This completes the proof of Theorem 2.6.

It is very likely that, if we use k-tuples in place of triples, the above argument can be modified to give a better lower bound on the number of unit distances determined by a point set.

Geometric graphs

3.1 Preliminaries

Graphs where the vertices are points in the plane in general position and the set of edges is a subset of the line segments between some pairs of the given points called *geometric graphs*. The investigation of geometric graphs is a very alive topic, there are many results about these graphs. On the other hand there are many unsolved, innocent-looking problems in this field. One of them is the following. What is the clique number (the size of the largest complete subgraph) of the intersection graph of a complete geometric graph on n points. With other words, what is the largest $f(n)$ such that if you have n points on the plane, then there are $f(n)$ pairs of points such that the line segments between the point pairs are pairwise crossing. Aronov, Erdős, Goddard, Kleitman, Klugerman, Pach and Schulman proved that $f(n)$ is $\Omega(\sqrt{n})$, but it is possible that there is a positive constant $c > 0$ such that $f(n) > cn$. In this chapter we characterize the complete subgraphs of the intersection graph and prove that at least the size of the largest complete bipartite graph is linear.

3.2 Halving lines

Let $P = \{p_1, p_2, \ldots, p_{2n}\}$ be a set of $2n$ points in the plane in general position, i.e., no three points are collinear. A line p_ip_j is said to be a *halving line* of P if both open half-planes bounded by p_ip_j contain precisely $n-1$ points. The number of halving lines of P is denoted by $h(P)$.

Taking an arbitrary line through any point of P and turning it around by at most 180 degrees, it always arrives at a position where it becomes a halving line. Thus, we have $h(P) \geq n$, and equality holds, e.g., when P is the vertex set of a convex $2n$-gon.

It is an intriguing open problem to determine the asymptotic behavior of $h(n) = \max_P h(P)$, where the maximum is taken over all $2n$-element sets in general position in the plane. It is known that

$$c_1 n e^{\sqrt{\log n}} \leq h(n) \leq c_2 n^{4/3}$$

for suitable constants $c_1, c_2 > 0$ (see [58],[88], [33], [21]). This function plays an important role in the analysis of many algorithms in computational geometry (cf. [28]).

We say that two segments *cross* if they have an interior point in common. Let $c(P)$ denote the maximum number of pairwise crossing segments $\overline{p_i p_j}$ whose endpoints belong to P. Obviously, $c(P) \leq n$ holds for every $2n$-element set P. If $c(P) = n$, we say that P has a *perfect cross-matching*. This is the case, for example, when P is the vertex set of a convex $2n$-gon.

Let $c(n) = \min_P c(P)$, where the minimum is taken over all $2n$-element sets in general position in the plane. We have

$$c(n) \geq c_3 \sqrt{n},$$

for some positive constant c_3, but there is no sublinear upper bound known for $c(n)$ (see [6],[68]). In fact, in [6] it was shown that every $2n$-element set in general position has an at least $c_3\sqrt{n}$-element subset which not only admits a perfect cross-matching, but also satisfies a much stronger condition. In this strong sense the result is best possible [91]. It looked difficult to improve the lower bound on $c(n)$, because we had no good characterization of perfectly cross-matchable sets.

The aim of this section is to give such a good characterization and to design an efficient algorithm which decides whether a set admits a perfect cross-matching.

3.2.1 Characterization of perfectly cross-matchable sets

In this section, we would like to point out a simple relation between $c(P)$ and $h(P)$: the first quantity attains its maximum if and only if the second attains its minimum. More precisely, we have the following.

Theorem 3.1 *A set of $2n$ points in general position in the plane admits a perfect cross-matching if and only if it has precisely n halving lines.*

Let me remark that the theorem follows from a recent result of Andrzejak, Aronov, Har-Peled, Seidel, and Welzl [4]. They proved a strong relation between the number of crossings of the halving segments and the incidence structure of segments and the points.

PROOF. Suppose first that $P = \{p_1, p_2, \ldots, p_{2n}\}$ has a perfect cross-matching (i.e., n pairwise crossing segments) $\overline{p_{2i-1}p_{2i}}, 1 \leq i \leq n$. The extension of each of these segments is a halving line, because each of them separates the two endpoints of all other segments $\overline{p_{2i-1}p_{2i}}$. We will show that P has no other halving lines.

Assume, in order to obtain a contradiction, that (say) p_1p_3 is also a halving line. We may suppose without loss of generality that p_1p_2 is horizontal, p_2 is to the right of p_1, and that p_{2i} is below and p_{2i-1} is above p_1p_2, for every $2 \leq i \leq n$. Since each segment $\overline{p_{2i-1}p_{2i}}\ \ (3 \leq i \leq n)$ has to cross $\overline{p_1p_2}$, if $\overline{p_{2i-1}p_{2i}}$ has an endpoint to the left of p_1p_3, then its other endpoint must lie to the right of p_1p_3. However, both p_2 and p_4 are on the right-hand side of p_1p_3. This implies that the number of points to the right of p_1p_3 exceeds by at least 2 the number of points to the left of it, contradicting our assumption that p_1p_3 is a halving line.

Suppose next that P has precisely n halving lines. Since there is *at least* one halving line through every point p_k, we obtain that there must be *exactly* one. Thus, we can assume without loss of generality that the complete list of halving lines is $p_{2i-1}p_{2i}\ \ (1 \leq i \leq n)$. We will show that the segments $\overline{p_{2i-1}p_{2i}}\ \ (1 \leq i \leq n)$ are pairwise crossing.

Assume, for contradiction, that $\overline{p_1p_2}$ and $\overline{p_3p_4}$ have no interior point in common. By renumbering the points if necessary, we may also suppose that p_1p_2 is horizontal, p_2 is to the right of p_1, $\overline{p_3p_4}$ is entirely above the line p_1p_2, and that p_3 is closer to p_1p_2 than p_4 is. Notice that a minor counter-clockwise turn around p_3 will bring the line $\ell = p_3p_4$ into a position, where there are precisely n points on its right-hand side. (Indeed, p_4 will be added to the set of points to the right of p_3p_4.) If we continue to turn ℓ around p_3 in the counter-clockwise direction, we arrive at a position where ℓ becomes parallel to p_1p_2, i.e., it becomes horizontal. At that moment, there are at most $n-2$ points above ℓ (these points form a subset of the set of all points different from p_3 which lie above the halving line p_1p_2). Hence, there is an intermediate position $\ell = p_3p_k$ for some $k \neq 4$, in which the number of points on the right-hand side of ℓ is precisely $n-1$. This means that p_3p_k is a halving line which does not appear in the complete list of halving lines, $p_{2i-1}p_{2i}\ \ (1 \leq i \leq n)$ which is a contradiction.

Actually, this argument also yields the uniqueness of the perfect cross-matching.

Theorem 3.2 *Any set of points in general position in the plane admits at most one perfect cross-matching.*

PROOF. As we have shown, every perfect cross-matching of P consists of exactly those segments between two points of P, whose extensions are *halving lines* of P.

3.2.2 Algorithm

The above characterization of perfectly cross-matchable sets allows us to design an $O(n \log n)$-time algorithm which decides whether a set of $2n$ points satisfies this property and, if so, finds a perfect cross-matching for it.

Let P be a $2n$-element point set in general position in the plane, which is the union of two n-element sets, P_1 and P_2, separated by a straight line (say, by the y-axis). For any non-vertical line ℓ, let $P_i(\ell^+)$ (resp. $P_i(\ell^-)$) denote the set of points in P_i lying above (resp. below) ℓ. A line ℓ not passing through any point of P is called a *ham-sandwich cut* for P, if

$$|P_1(\ell^+)| = |P_2(\ell^-)| = \lfloor n/2 \rfloor.$$

It was shown by Megiddo [61] that one can always find such a line ℓ in $O(n)$ steps (see also [57]).

Any matching of P that has a segment to the left of the y-axis, has another one to the right of it, and these two segments cannot cross. Thus, if there exists a perfect cross-matching for P, then all of its segments must cross the y-axis and, similarly, they must also cross the ham-sandwich cut ℓ. Consequently, a perfect cross-matching M for P is the union of a perfect cross-matching M_1 for $P_1(\ell^+) \cup P_2(\ell^-)$ and a perfect cross-matching M_2 for $P_1(\ell^-) \cup P_2(\ell^+)$. Let M_i^+ and M_i^- denote the *upper envelope* and the *lower envelope* (i.e., the pointwise maximum and pointwise minimum) of the lines supporting the segments of M_i, respectively ($i = 1, 2$). Clearly, M_i^+ and M_i^- are unbounded convex polygonal paths, with at most $\lfloor n/2 \rfloor$ vertices each. (See Figure 3.1)

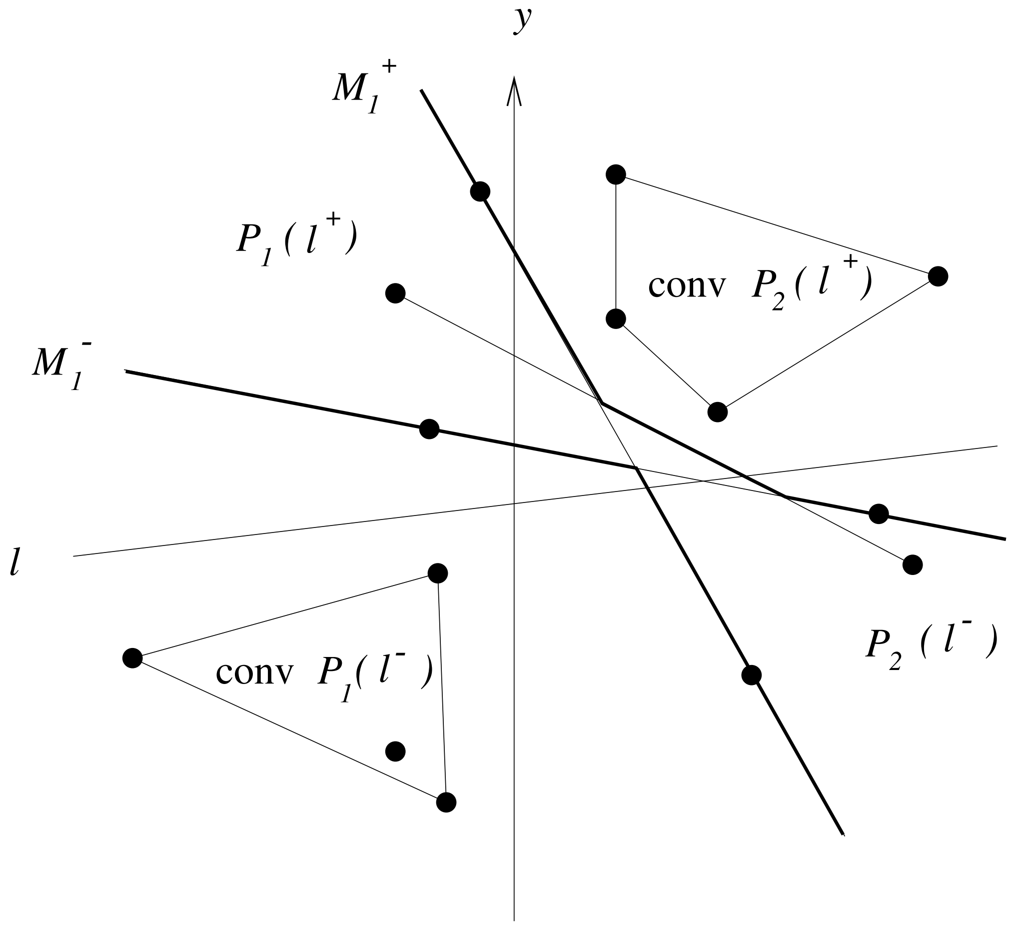

Figure 3.1

We need the following corollary of Theorem 3.2.

Observation 3.1 *The set P admits a perfect cross-matching M if and only if the following conditions are satisfied.*

(1) $P_1(\ell^+) \cup P_2(\ell^-)$ *admits a perfect cross-matching* M_1 *and* $P_1(\ell^-) \cup P_2(\ell^+)$ *admits a perfect cross-matching* M_2.

(2) *The convex hull* $\mathrm{conv} P_2(\ell^+)$ *is above the polygonal path* M_1^+, *and* $\mathrm{conv} P_1(\ell^-)$ *is below* M_1^-. *Similarly,* $\mathrm{conv} P_1(\ell^+)$ *is above* M_2^+, *and* $\mathrm{conv} P_2(\ell^-)$ *is below* M_2^-.

Then, we have $M = M_1 \cup M_2$.

Proof. We have seen before that if P admits a perfect cross-matching M, then it satisfies condition (1) and $M = M_1 \cup M_2$ holds. By Theorem 3.2, M_1 and M_2 are uniquely determined. To see that (2) is necessary, too, assume that (say) $P_2(\ell^+)$ has a point p below M_1^+. Then p lies below the supporting line of at least one segment $\overline{qq'} \in M_1$. Let p' denote the element of $P_1(\ell^-)$

connected to p in M_2. Then $\overline{pp'} \cap \overline{qq'} = \emptyset$, contradicting our assumption that any two segments of M cross.

Suppose next that conditions (1) and (2) are satisfied. Then $M = M_1 \cup M_2$ is a perfect cross-matching for P. Indeed, if there were two disjoint segments, $\overline{pp'} \in M_1$ and $\overline{qq'} \in M_2$, such that (say) $\overline{qq'}$ is below (resp. above) the line pp', then $\mathrm{conv}P_2(\ell^+)$ would not lie above the polygonal path M_1^+ (resp. $\mathrm{conv}P_1(\ell^-)$ would not lie below M_1^-), contradicting condition (2).

Let M^+ and M^- denote the upper and the lower envelope of all lines supporting the segments of $M = M_1 \cup M_2$, respectively. Clearly, M^+ can be obtained as the upper envelope of M_1^+ and M_2^+, and M^- can be obtained as the lower envelope of M_1^- and M_2^-.

It is well known that one can compute the union and the intersection of two convex polygons of at most n sides in time $O(n)$ ([74], [78]). Thus, if we know $\mathrm{conv}P_i(\ell^+), \mathrm{conv}P_i(\ell^-), M_i^+$, and M_i^- for $i = 1, 2$, then in linear time we can determine $\mathrm{conv}P_i(i = 1, 2), M^+$ and M^-. If any of the conditions of the Claim is not satisfied, we conclude that P does not admit a perfect cross-matching.

So one can use a divide-and-conquer algorithm to decide whether $P = P_1 \cup P_2$ admits a perfect cross-matching and, if yes, to compute it simultaneously with $\mathrm{conv}P_i \quad (i = 1, 2), M^+$ and M^-. At each stage it takes linear time to find a ham-sandwich cut ℓ and to do the merge step.

We obtained the following.

Theorem 3.3 *There is an $O(n \log n)$ time, $O(n)$ space algorithm which decides whether a set of $2n$ points in general position in the plane admits a perfect cross-matching and, if so, computes it.*

Clearly, any decision tree that determines the perfect cross-matching of a planar point set of $2n$ points (if it exists) has height $\Omega(n \log n)$. In this sense this theorem is asymptotically tight.

3.3 Large crossing pairs

In this section we prove that if a geometric graph is dense then the intersection graph of the edges contains a large complete bipartite subgraph. In all of these results, A stands for an absolute constant smaller than 10^6. We shall see the origin of this number in the next chapter. Also the proofs are based on results which will be proved in the next chapter.

Theorem 3.4 *Any geometric graph G with n vertices and at least cn^2 edges $(c > 0)$ has two disjoint sets of edges $E_1, E_2 \subset E(G)$ such that $|E_1|, |E_2| \geq (c/32)^{A+3}\binom{n}{2}$ and every edge in E_1 crosses all edges in E_2.*

In our previous result about different distances, we used a result of Székely about the crossing number of geometric graphs. Here we recall an earlier result from this field. Let G be a geometric graph with n vertices and at least cn^2 edges. The next result by Ajtai, Chvátal, Newborn, Szemerédi [1] and, independently, Leighton [60] (see also [69, 72]) implies that there are at least $\frac{c}{64}e^2$ crossing pairs of edges.

Lemma 3.1 *Let G be a geometric graph with n vertices and $e > 4n$ edges, for some $c > 0$. Then G has at least $\frac{e^3}{64n^2}$ crossing pairs of edges.*

As we saw in the first chapter, every dense bipartite graph contains complete bipartite subgraphs. More accurately the theorem of Kővári, Sós, and Turán [50] states that every graph with n vertices and at least cn^2 edges has a complete bipartite subgraph with $c' \log n$ vertices in its classes, where $c' > 0$ is a suitable constant depending on c. This immediately implies that Theorem 3.4 holds with the much weaker bound $c' \log n$ instead of $c'n$.

Theorem 3.5 *Any geometric graph G with n vertices and at least cn^2 edges $(c > 0)$ has two disjoint sets of edges $E_1, E_2 \subset E(G)$ such that $|E_1|, |E_2| \geq (c/34)^{A+3}\binom{n}{2}$ and no edge in E_1 crosses any edge in E_2.*

Lemma 3.2 *[67] Let G be a geometric graph with n vertices and $e \geq 3n/2$ edges, for some $c > 0$. Then G has at least $\frac{4e^3}{27n^2}$ pairs of edges that do not cross and do not share an endpoint.*

The above theorems can also be established using Szemerédi's Regularity Lemma [85]. However, then the dependence on c of the sizes of the homogeneous subsystems whose existence is guaranteed by our results would get much worse.

For the proofs one should simply apply the above two Lemmata and Theorem 4.4 or Theorem 4.5, respectively.

Line segments

4.1 Preliminaries

Given a system $\mathcal{S}$ of simple continuous curves ("strings") in the plane, we can define a graph $G_{\mathcal{S}}$ as follows. Assign a vertex to each curve, and connect two vertices by an edge if and only if the corresponding two curves intersect. $G_{\mathcal{S}}$ is called the *intersection graph* of $\mathcal{S}$.

Not every graph is an intersection graph of a system of curves [25] (see Figure 4.1 for a simple example). This implies (see [32] and Chapter 5 for details) that only a very small fraction of all the $2^{\binom{n}{2}}$ labeled graphs on n vertices have this property. For systems of *segments* we obtain a fairly good quantitative result.

Theorem 4.1 *The number of labeled graphs on n vertices which can be obtained as the intersection graph of a system of segments in the plane is $2^{O(n \log n)}$.*

The problem of recognizing intersection graphs of planar curves (the so-called "string graph problem") is known to be NP-hard [51], but it is open whether this problem is *decidable* [52]. In some very special cases, e.g., when $\mathcal{S}$ consists of segments, there are trivial recognition algorithms [16], [38]. But even in these cases we do not know much about the structure of intersection graphs. One of the most striking examples illustrating our ignorance in the subject is the following simple open problem.

Problem 4.1 *Is it true that every planar graph is the intersection graph of a system of segments in the plane?*

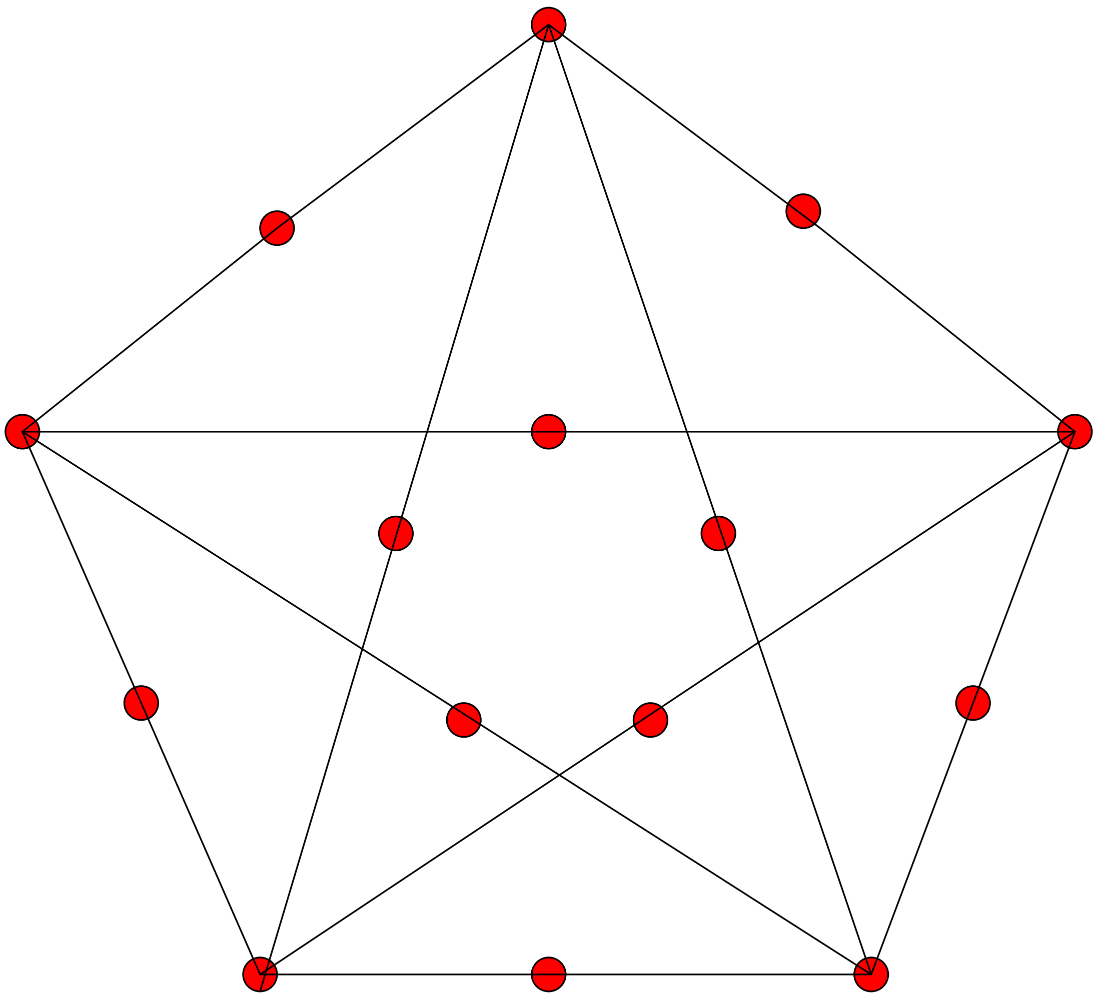

Figure 4.1 A graph which is *not* the intersection graph of any system of segments. This would contradict planarity of K_5.

The aim of this chapter is to prove some Ramsey-type results for intersection graphs of segments. In other words, we establish *necessary* conditions for a graph to be the intersection graph of a system of segments. We recall a simple observation from Ramsey theory. As usual, let $V(G)$ and $E(G)$ denote the vertex set and the edge set of a graph G, respectively.

Theorem 4.2 *[45] Let H be a fixed graph of k vertices. Then every graph G with n vertices, which does not contain an induced subgraph isomorphic to H, has two disjoint sets of vertices, $V_1, V_2 \subset V(G)$, such that $|V_1|, |V_2| \geq n^{1/(k-1)}/2$ and*

(i) either all edges between V_1 and V_2 belong to G,

(ii) or no edge between V_1 and V_2 belongs to G.

Note that the weaker result, with roughly $\log n$ in the place of $n^{1/(k-1)}$, immediately follows from Ramsey's theorem [35]. Combining the last theorem with Theorem 4.1 (or rather with the fact that there is *at least one* forbidden induced subgraph in the class of all segment intersection graphs, say, the 15 vertex graph depicted in Figure 4.1), we obtain the following

Corollary 4.3 *There exists a constant $\varepsilon \geq 1/14$ such that every system $\mathcal{S}$ of n segments in the plane has two disjoint subsystems $\mathcal{S}_1, \mathcal{S}_2 \subset \mathcal{S}$ such that $|\mathcal{S}_1|, |\mathcal{S}_2| \geq n^{\varepsilon}/2$ and*

(i) either every segment in $\mathcal{S}_1$ crosses all segments in $\mathcal{S}_2$,

(ii) or no segment in $\mathcal{S}_1$ crosses any segment in $\mathcal{S}_2$.

Note that here, as well as in the sequel, we only consider systems of segments in *general position*, i.e., we assume that no two segments are parallel and no three endpoints are collinear.

The main result of this chapter, formulated in the next two statements, substantially strengthens Corollary 4.3. In all of these results, A stands for an absolute constant smaller than 10^6.

Theorem 4.4 *Any system $\mathcal{S}$ of n segments in the plane with at least cn^2 crossings $(c > 0)$ has two disjoint subsystems, $\mathcal{S}_1, \mathcal{S}_2 \subset \mathcal{S}$, such that $|\mathcal{S}_1|, |\mathcal{S}_2| \geq \frac{(2c)^A}{660} n$ and every segment in $\mathcal{S}_1$ crosses all segments in $\mathcal{S}_2$.*

Theorem 4.5 *Any system $\mathcal{S}$ of n segments in the plane with at least cn^2 non-crossing pairs $(c > 0)$ has two disjoint subsystems, $\mathcal{S}_1, \mathcal{S}_2 \subset \mathcal{S}$, such that $|\mathcal{S}_1|, |\mathcal{S}_2| \geq \frac{(c/5)^A}{330} n$ and no segment in $\mathcal{S}_1$ crosses any segment in $\mathcal{S}_2$.*

4.2 Bounding the number of intersection graphs of segments

The aim of this section is to prove Theorem 4.1.

Let $\mathcal{S} = \{s_1, s_2, \ldots, s_n\}$ be a system of segments in general position in the plane. Assume that s_i is not parallel to the y-axis and can be described by the relations:

$$s_i : \quad y = a_i x + b_i,\ c_i \le x \le d_i \quad (i = 1, 2, \ldots, n).$$

Two segments, s_i and s_j, cross each other if and only if

$$\max\{c_i, c_j\} \le \frac{b_j - b_i}{a_i - a_j} \le \min\{d_i, d_j\}.$$

Thus, whether or not s_i crosses s_j, is determined by the sign of polynomials

$$P_{i,j} := a_i - a_j, \quad Q_{i,j} := c_i(a_i - a_j) + b_i - b_j, \quad R_{i,j} := d_i(a_i - a_j) + b_i - b_j,$$

for any $1 \le i < j \le n$. These are $6\binom{n}{2}$ polynomials (for ordered i, j pairs) of degree at most 2 in the $4n$ variables $\{a_i, b_i, c_i, d_i\}$.

We use the following form of the Milnor-Thom theorem [62, 87].

Theorem. [92] *The number of different sign patterns for m polynomials of degree at most d in k variables is at most $(4edm/k)^k$.*

Applying this bound with $m = 6\binom{n}{2}$, $d = 2$, and $k = 4n$, we obtain that there are at most $(6e(n-1))^{4n} = 2^{O(n \log n)}$ different ways how the crossing relationship can be defined among n segments in the plane.

4.3 Homogeneous Pairs

Three sets of points in the plane are said to be *separable* if each of them can be separated from the other two by a straight line. Given three separable sets, there is no straight line which intersects the convex hull of all of them.

Lemma 4.1 *Every set of n points in general position in the plane has three separable subsets of size $\lfloor n/6 \rfloor$.*

PROOF. Assume without loss of generality that n is divisible by 6, and let P be an n-element point set. Choose two lines that divide the plane into 4 regions, containing $m, 2m, m,$ and $2m$ points of P in their interiors, in this cyclic order. Let $P_1, P_2, P_3,$ and P_4 denote the corresponding subsets of P. By the *ham-sandwich theorem*, there is a line ℓ which simultaneously cuts P_2 and P_4 into two halves of equal size (see Figure 4.2). Then ℓ avoids either the convex hull of P_1 or that of P_3. Assume, by symmetry, that P_1 is 'above' ℓ. Then P_1 and the parts of P_2 and P_4 'below' ℓ are three separable sets.

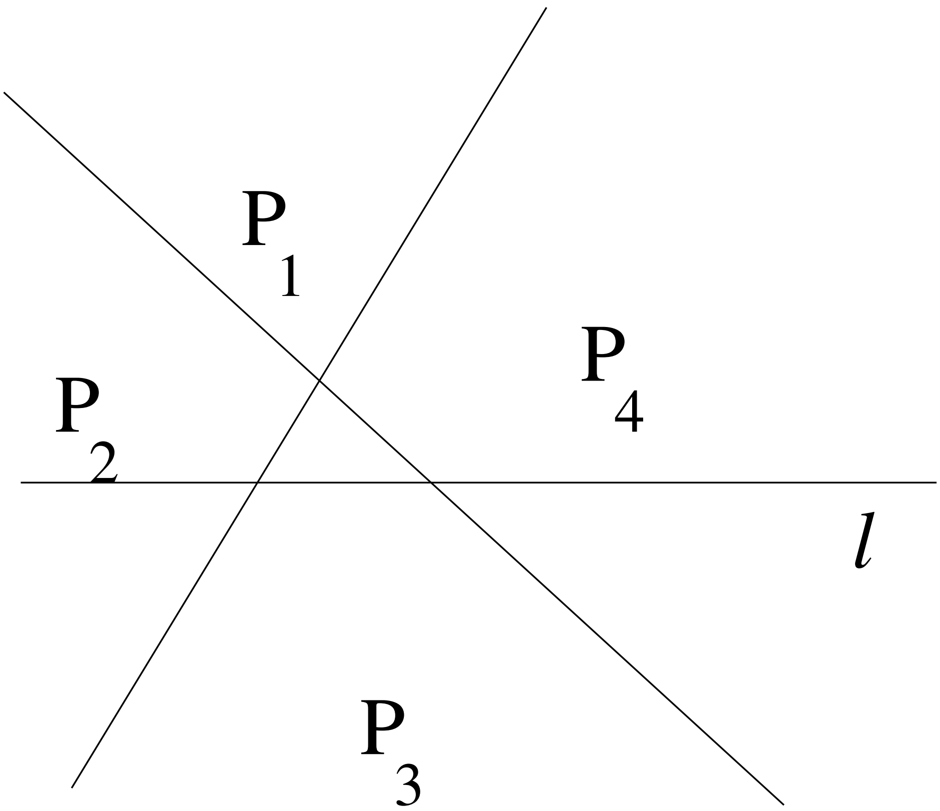

Figure 4.1

Lemma 4.2 *Let $\mathcal{S}$ and $\mathcal{T}$ be two systems of segments in general position in the plane. Then there are two subsystems $\mathcal{S}^* \subseteq \mathcal{S}$, $\mathcal{T}^* \subseteq \mathcal{T}$ such that $|\mathcal{S}^*| \geq \lfloor |\mathcal{S}|/330 \rfloor$, $|\mathcal{T}^*| \geq \lfloor |\mathcal{T}|/330 \rfloor$, and*

(i) either every segment in $\mathcal{S}^*$ crosses all segments in $\mathcal{T}^*$,

(ii) or no segment in $\mathcal{S}^*$ crosses any segment in $\mathcal{T}^*$.

PROOF. Let $|\mathcal{S}| = m, |\mathcal{T}| = n$, and suppose, for simplicity, that both m and n are multiples of 330. Let P be the set of endpoints of all segments in $\mathcal{S}$. By Lemma 4.1, there are three separable $m/3$-element subsets, $P_1, P_2, P_3 \subseteq P$. Color a segment $t \in \mathcal{T}$ with color i if its supporting line does not intersect the convex hull of P_i $(i = 1, 2, 3)$. Let $\mathcal{T}_i$ denote the segments of color i. At least one third of the elements of $\mathcal{T}$ get the same color, so we can assume with no loss of generality that $|\mathcal{T}_1| \geq n/3$.

If there are at least $m/330$ segments in $\mathcal{S}$, both of whose endpoints belong to P_1, then we are done, because these segments are disjoint from all elements of $\mathcal{T}_1$.

Hence, we can assume that at least $(1/3 - 2/330)m = 18m/55$ elements of $\mathcal{S}$ have precisely one of their endpoints in P_1. Let Q denote the set of other endpoints of these segments. Let us choose three separable subsets $Q_1, Q_2, Q_3 \subseteq Q$, each of size at least $|Q|/6 = 3m/55$. Just as before, color a segment $t \in \mathcal{T}_1$ with color i if its supporting line does not intersect the convex hull of Q_i $(i = 1, 2, 3)$. Again, at least $|\mathcal{T}_1|/3 \geq n/9$ elements of $\mathcal{T}_1$ get the same color, say color 1; they form a subsystem $\mathcal{T}_{11} \subseteq \mathcal{T}_1$.

Let $\mathcal{S}_{11}$ denote the set of all elements of $\mathcal{S}$ with one endpoint in P_1 and the other in Q_1. Clearly, we have $|\mathcal{S}_{11}| = |Q_1| \geq 3m/55$.

Let us repeat now the whole procedure with $\mathcal{T}_{11}$ in the place of $\mathcal{S}$ and $\mathcal{S}_{11}$ in the place of $\mathcal{T}$. We obtain two subsets, $\mathcal{T}' \subseteq \mathcal{T}_{11}$ and $\mathcal{S}' \subseteq \mathcal{S}_{11}$, satisfying

$$|\mathcal{T}'| \geq \frac{3|\mathcal{T}_{11}|}{55} \geq \frac{n}{165}, \quad |\mathcal{S}'| \geq \frac{|\mathcal{S}_{11}|}{9} \geq \frac{m}{165}.$$

We can assume that at least half of the supporting lines of the elements of $\mathcal{T}'$ cross the convex hull of $\mathcal{S}'$, for otherwise we would obtain two non-crossing systems of at least $|\mathcal{T}'|/2$ and $|\mathcal{S}'|$ segments. The set of all elements of $\mathcal{T}'$, whose supporting lines cross the convex hull of $\mathcal{S}'$ is denoted by $\mathcal{T}^*$. Similarly, we can assume that the supporting lines of at least half of the elements of $\mathcal{S}'$ cross the convex hull of $\mathcal{T}^*$; otherwise, we could find two non-crossing systems of at least $|\mathcal{T}^*|$ and $|\mathcal{S}'|/2$ segments. Let $\mathcal{S}^*$ denote the set of all elements of $\mathcal{S}'$, whose supporting lines cross the convex hull of $\mathcal{T}^*$. It follows from the selection of $\mathcal{S}^*$ that every element of $\mathcal{S}^*$ crosses all elements of $\mathcal{T}^*$ and that

$$|\mathcal{S}^*| \geq \frac{|\mathcal{S}|'}{2} \geq \frac{m}{330}, \quad |\mathcal{T}^*| \geq \frac{|\mathcal{T}|'}{2} \geq \frac{n}{330}.$$

Given any system of segments, $\mathcal{S}$ and $\mathcal{T}$, in general position in the plane, define their *crossing density*, $\delta(\mathcal{S}, \mathcal{T})$, as the number of crossing pairs $(s, t),\ s \in \mathcal{S},\ t \in \mathcal{T}$ divided by $|\mathcal{S}| \cdot |\mathcal{T}|$. Clearly, we have $0 \leq \delta(\mathcal{S}, \mathcal{T}) \leq 1$.

Theorem 4.6 *There exists a constant $A < 10^6$ satisfying the following condition. Let $\mathcal{S}$ and $\mathcal{T}$ be any sets of segments in general position in the plane, and suppose that their crossing density is at least $c > 0$. Then there are two disjoint subsystems $\mathcal{S}' \subseteq \mathcal{S}$, $\mathcal{T}' \subseteq \mathcal{T}$ such that*

$$|\mathcal{S}'| \geq \frac{c^A}{330}|\mathcal{S}|, \ \ |\mathcal{T}'| \geq \frac{c^A}{330}|\mathcal{T}|,$$

and every segment in $\mathcal{S}'$ crosses all segments in $\mathcal{T}'$.

PROOF. Let $|\mathcal{S}| = m, |\mathcal{T}| = n$, and suppose first that both m and n are powers of 330. According to our assumption, $\delta(\mathcal{S}, \mathcal{T}) \geq c$.

Applying Lemma 4.2, we obtain two subsystems, $\mathcal{S}^* \subset \mathcal{S}$, $\mathcal{T}^* \subset \mathcal{T}$, such that $|\mathcal{S}^*| = m/330$, $|\mathcal{T}^*| = n/330$, and $\delta(\mathcal{S}^*, \mathcal{T}^*)$ is either 1 or 0. In the first case we are done, so assume $\delta(\mathcal{S}^*, \mathcal{T}^*) = 0$. Then we have

$$c \leq \delta(\mathcal{S}, \mathcal{T}) =$$

$$\frac{329}{330^2}\delta(\mathcal{S}, \mathcal{T} - \mathcal{T}^*) + \frac{329}{330^2}\delta(\mathcal{S} - \mathcal{S}^*, \mathcal{T}) + \frac{329^2}{330^2}\delta(\mathcal{S} - \mathcal{S}^*, \mathcal{T} - \mathcal{T}^*).$$

Therefore, at least one of the crossing densities $\delta(\mathcal{S}, \mathcal{T} - \mathcal{T}^*)$, $\delta(\mathcal{S} - \mathcal{S}^*, \mathcal{T})$, $\delta(\mathcal{S} - \mathcal{S}^*, \mathcal{T} - \mathcal{T}^*)$ exceeds

$$c_1 := c\frac{330^2}{330^2 - 1}.$$

In other words, there exist two subsystems, $\mathcal{S}_1 \subset \mathcal{S}$, $\mathcal{T}_1 \subset \mathcal{T}$, with $|\mathcal{S}_1| \geq m/330$, $|\mathcal{T}_1| \geq n/330$ such that $\delta(\mathcal{S}_1, \mathcal{T}_1) \geq c_1$.

Applying Lemma 4.2 to $\mathcal{S}_1$ and $\mathcal{T}_1$, we obtain two subsystems $\mathcal{S}^{**} \subset \mathcal{S}_1$, $\mathcal{T}^{**} \subset \mathcal{T}_1$, such that $|\mathcal{S}^{**}| \geq m/330^2, |\mathcal{T}^{**}| \geq n/330^2$, and $\delta(\mathcal{S}^{**}, \mathcal{T}^{**})$ is either 1 or 0. Again, we can assume that $\delta(\mathcal{S}^{**}, \mathcal{T}^{**}) = 0$, otherwise we are done. As before, we can find two subsystems, $\mathcal{S}_2 \subset \mathcal{S}_1$, $\mathcal{T}_2 \subset \mathcal{T}_1$, with $|\mathcal{S}_2| \geq m/330^2$, $|\mathcal{T}_2| \geq n/330^2$ such that

$$\delta(\mathcal{S}_2, \mathcal{T}_2) \geq c_2 := c\left(\frac{330^2}{330^2 - 1}\right)^2.$$

Since the crossing density between any two sets is at most 1, after

$$k \leq \frac{\log \frac{1}{c}}{\log \frac{330^2}{330^2-1}}$$

steps, this procedure will terminate. That is, when we apply Lemma 4.2 for the k-th time, we obtain two subsystems $\mathcal{S}' \subseteq \mathcal{S}$, $\mathcal{T}' \subseteq \mathcal{T}$ such that $|\mathcal{S}'| \geq m/330^k$, $|\mathcal{T}'| \geq n/330^k$, and $\delta(\mathcal{S}', \mathcal{T}') = 1$. Thus, every element of $\mathcal{S}'$ crosses all elements of $\mathcal{T}'$, and $|\mathcal{S}'| \geq c^A m,\ |\mathcal{T}'| \geq c^A n$, where

$$A \leq \frac{\log 330}{\log \frac{330^2}{330^2-1}} < 10^6.$$

This completes the proof of Theorem 4.3 in the case when m and n are powers of 330. Otherwise, using an easy averaging argument, we can find $\mathcal{S}_0 \subseteq \mathcal{S}$, $\mathcal{T}_0 \subseteq \mathcal{T}$, whose sizes are powers of 330, $|\mathcal{S}_0| \geq m/330$, $|\mathcal{T}_0| \geq n/330$, and $\delta(\mathcal{S}_0, \mathcal{T}_0) \geq c$. Applying the above argument to $\mathcal{S}_0$ and $\mathcal{T}_0$, the result follows.

PROOF. (of Theorem 4.4) Assume, for simplicity, that n is even. Given a system of n segments in general position in the plane, which determine at least cn^2 crossings, one can partition it into two equal parts so that the crossing density between them is at least $2c$ (see e.g. [69]). Applying Theorem 4.6 to these parts, the result follows.

Theorem 4.5 can be established analogously, by repeated application of Lemma 4.2. However, here we deduce it from Theorems 4.4 and 4.6.

PROOF. (of Theorem 4.5) Let $\mathcal{S}$ be a set of n segments in general position in the plane with at least cn^2 non-crossing pairs. For any $s \in \mathcal{S}$, let $\ell(s)$ denote the supporting line of s. The set $\ell(s) \setminus s$ consists of two half-lines; denote them by $h_1(s)$ and $h_2(s)$. Let $\mathcal{H}_1 := \{h_1(s) : s \in \mathcal{S}\}$, $\mathcal{H}_2 := \{h_2(s) : s \in \mathcal{S}\}$, $\mathcal{T} := \mathcal{S} \cup \mathcal{H}_1 \cup \mathcal{H}_2$. Further, for any $h \in \mathcal{H}_1 \cup \mathcal{H}_2$, let $s(h)$ be the unique segment $s \in \mathcal{S}$, for which $h_1(s)$ or $h_2(s)$ is equal to s.

Note that if two segments $s, t \in \mathcal{S}$ do not cross each other, then the crossing between their supporting lines, $\ell(s)$ and $\ell(t)$, gives rise to a crossing between a pair of elements of $\mathcal{T}$, involving at least one half-line. Therefore, the number of crossing pairs in $\mathcal{T}$ involving at least one half-line is at least cn^2. There are three possibilities:

1. for some $i = 1, 2$, the number of crossing pairs in $\mathcal{H}_i$ is at least $cn^2/5$;
2. the number of crossing pairs between $\mathcal{H}_1$ and $\mathcal{H}_2$ is at least $cn^2/5$;
3. for some $i = 1, 2$, the number of crossing pairs between $\mathcal{H}_i$ and $\mathcal{S}$ is at least $cn^2/5$.

In Case 1, applying Theorem 4.4 to $\mathcal{H}_i$, we obtain two subsystems, $\mathcal{H}_{i1}, \mathcal{H}_{i2} \subset \mathcal{H}$, whose sizes are at least $\frac{(2c/5)^A}{660}n > \frac{(c/5)^A}{330}$, and every half-line in $\mathcal{H}_{i1}$ crosses all half-lines in $\mathcal{H}_{i2}$. Then $\mathcal{S}_1 := \{s(h) : h \in \mathcal{H}_{i1}\}$ and $\mathcal{S}_2 := \{s(h) : h \in \mathcal{H}_{i2}\}$ meet the requirements in Theorem 4.3.

In Case 2, apply Theorem 4.6 to obtain $\mathcal{H}'_1 \subseteq \mathcal{H}_1, \mathcal{H}'_2 \subseteq \mathcal{H}_2$, whose sizes are at least $\frac{(c/5)^A}{330}n$, and every element of $\mathcal{H}'_1$ crosses all elements of $\mathcal{H}'_2$. Setting $\mathcal{S}_1 := \{s(h) : h \in \mathcal{H}'_1\}$, and $\mathcal{S}_2 := \{s(h) : h \in \mathcal{H}'_2\}$, the result follows. Case 3 can be treated similarly.

4.4 Regular partition of line segments

At the end of this chapter we propose a fast approximate solution for reporting *most* intersections among n segments in the plane. This is not an explicit algorithm it is just about the existence of a "regular" partition. On the other hand I believe that there will be an efficient algorithm to find such a partition.

The problem of detecting and reporting intersections among straight-line segments in the plane is one of the oldest and most extensively studied topics in computational geometry. It is a basic ingredient of many hidden surface removal algorithms, and has numerous other applications in computer graphics, motion planning, geographic information systems, etc. The first efficient techniques were developed by Shamos and Hoey [79] and Bentley and Ottmann [10] more than twenty years ago. The running times of the best known algorithms, due to Balaban [7] and Chazelle and Edelsbrunner [15], are $O(n \log n + I)$, where n and I denote the number of segments and the number of intersections, resp. (See also [70].)

The results of this chapter, combined with Szemerédi's Regularity Lemma [85], can be used to establish a fairly strong structure theorem for intersection graphs of segments. We say that two sets have *almost the same* number of elements if their sizes differ by at most a factor of 2.

Theorem 4.7 *For any $\epsilon > 0$, there exists an integer $K = K(\epsilon)$ with the property that any system $\mathcal{S}$ of segments in the plane can be partitioned into $K+1$ subfamilies, $\mathcal{S}_0, \mathcal{S}_1, \ldots, \mathcal{S}_K$ such that $|\mathcal{S}_0| < \epsilon|\mathcal{S}|$, all other subfamilies have almost the same size, and for all but at most ϵK^2 pairs $1 \leq i, j \leq K$,*

(i) either every segment in $\mathcal{S}_i$ crosses all segments in $\mathcal{S}_j$,

(ii) or no segment in $\mathcal{S}_i$ crosses any segment in $\mathcal{S}_j$.

Fix an element s_i in each $\mathcal{S}_i$. For any $s \in \mathcal{S}$, let $f(s) := s_i$ if and only if s belongs $\mathcal{S}_i$ $(0 \leq i \leq K)$. We can think of $f(s)$ as the segment *representing* s. According to Theorem 4.7 , with a very small error, two randomly selected elements $s, t \in \mathcal{S}$ cross each other if and only if $f(s) \cap f(t)$ is non-empty.

The proof is based on a variant of Szemerédi's Regularity Lemma, which was discovered by Komlós (see [49]) and can be established by an elegant argument.

For any graph G and for any disjoint subsets $X, Y \subset V(G)$, let $E(X, Y) \subseteq E(G)$ denote the set of edges of G running between X and Y. Clearly, we have $|E(X, Y)| \leq |X||Y|$. For any $\gamma, \delta > 0$, we call the pair (X, Y) (γ, δ)-

superregular if for every $X' \subseteq X$ and $Y' \subseteq Y$ satisfying

$$|X'| \geq \gamma|X|, \quad |Y'| \geq \gamma|Y|,$$

we have

$$|E(X', Y')| \geq \delta|X||Y|.$$

Lemma 4.3 ([49]) *Let $\gamma > 0$ be a sufficiently small constant, and let $\delta > 0$. Then any graph with n vertices and at least δn^2 edges has a (γ, δ)-superregular pair (X, Y) with*

$$|X| = |Y| \geq \delta^{1/\gamma^2} n.$$

First we establish

Theorem 4.8 *For every $\delta > 0$, there exists an integer $k = k(\delta) > 0$ with the following property. The intersection graph G of any system of n segments in the plane has k bipartite subgraphs (not necessarily edge-disjoint), which altogether cover all but at most δn^2 edges of G.*

PROOF. Set $G_0 := G$, and let β be a small positive constant to be specified later. Suppose that for some $i \geq 1$ we have already defined G_{i-1}. If G does not have a complete bipartite subgraph H_i, which contains at least βn^2 edges of G_{i-1}, then stop. Otherwise, pick such a subgraph H_i, and let G_i denote the graph obtained from G_{i-1} by the deletion of all edges belonging to H_i. Obviously, this procedure will terminate in

$$j \leq \frac{|E(G)|}{\beta n^2} \leq \frac{1}{2\beta}$$

steps, with a graph G_j.

We claim that G_j has fewer than δn^2 edges, provided that β is sufficiently small. Suppose that this is not true. Then, according to Lemma 4.3, G_j has a (γ, δ)-superregular pair (X, Y) with

$$|X| = |Y| \geq \delta^{1/\gamma^2} n,$$

where $\gamma := \frac{\delta^A}{330}$. Let $\mathcal{S}_\mathcal{X}$ and $\mathcal{S}_\mathcal{Y}$ denote the corresponding families of segments. In view of Theorem 4.3, there are two disjoint subsystems $\mathbb{T}_X \subseteq \mathcal{S}_\mathcal{X}$ and $\mathbb{T}_Y \subseteq \mathcal{S}_\mathcal{Y}$ with

$$|\mathbb{T}_X| = |\mathbb{T}_Y| \geq \frac{\delta^A}{330}|X| = \gamma|X| = \gamma|Y|$$

such that every segment in $\mathbb{T}_X$ crosses all elements of $\mathbb{T}_Y$. Let X' and Y' denote the subsets of X and Y, corresponding to $\mathbb{T}_X$ and $\mathbb{T}_Y$, resp. Then X' and Y' induce a complete bipartite subgraph in G. Furthermore, using the fact that (X, Y) is a (γ, δ)-superregular pair in G_j, we obtain that at least

$$\delta|X'||Y'| \geq \delta\gamma^2\delta^{2/\gamma^2}n^2$$

edges between X' and Y' belong to G_j. Therefore, if we choose β so small that this last quantity exceeds βn^2, then we could continue our procedure and define the next graph G_{j+1}. This contradiction completes the proof.

Obviously, a similar result holds for $\overline{G}$, the *complement* of a segment intersection graph G.

Theorem 4.9 *For every $\delta > 0$, there exists an integer $k = k(\delta) > 0$ with the following property. The complement $\overline{G}$ of the intersection graph G of any system of n segments in in the plane has k bipartite subgraphs, which altogether cover all but at most δn^2 edges of $\overline{G}$.*

Now we are in a position to prove Theorem 4.7 Let G denote the intersection graph of $\mathcal{S}$. Let δ be a small positive constant which will be specified later. By Theorems 4.8 and 4.9, there is a family

$$\mathcal{F} = \{A_1, B_1, A_2, B_2, \ldots, A_{2k}, B_{2k}\}$$

of subsets of $V(G)$ such that

1. A_i and B_i are disjoint $(1 \leq i \leq 2k)$;
2. $A_i \times B_i$ is contained either in $E(G)$ or in $E(\overline{G})$ $\ (1 \leq i \leq 2k)$;
3. all but at most $2\delta n^2$ pairs $\{u, v\} \subset V(G)$ are covered by $\cup_{i=1}^{2k} A_i \times B_i$.

We say that two vertices of $V(G)$ are of the *same type*, if every member of $\mathcal{F}$ contains both or neither of them. The number of different types is at most 3^{2k}. A given type is *negligible*, if fewer than

$$s := \frac{\epsilon n}{3^{2k}}$$

vertices have it. Letting V_0 denote the set of all vertices with negligible types, we have $|V_0| < \epsilon n$.

Divide the elements of $V(G) - V_0$ into groups $V_1, V_2, \ldots, V_K$ such that all elements are from the same type and the groups have almost the same size: for every $1 \le i \le K$, let $s \le |V_i| \le 2s$. Clearly, we have

$$\frac{(1-\epsilon)n}{2s} \le K \le \frac{n}{s}.$$

A pair $(i,j),\ 1 \le i \ne j \le K$ is called *exceptional*, if $V_i \times V_j$ is not contained in $\cup_{i=1}^{2k} A_i \times B_i$. For every non-exceptional pair (i,j), V_i and V_j induce a complete bipartite subgraph either in G or in $\overline{G}$.

Let m denote the number of exceptional pairs. The total number of pairs $\{u,v\} \subset V(G)$ for which $u \in V_i,\ v \in V_j$ for some exceptional pair (i,j) is at least ms^2. On the other hand, by condition 3 above, this number cannot exceed $2\delta n^2$. Thus, we obtain that

$$\frac{m}{K^2} \le \frac{2\delta n^2}{s^2 K^2} \le \frac{2\delta n^2}{s^2} \frac{4s^2}{(1-\epsilon)^2 n^2} = \frac{8\delta}{(1-\epsilon)^2}.$$

This is smaller than ϵ, if δ is sufficiently small, so the partition of $\mathcal{S}$ corresponding to $V_0 \cup V_1 \cup \ldots \cup V_K$ meets all the requirements of Theorem 4.7.

Tools from graph theory

5.1 Introduction

There are several examples when graph theoretical results help to prove statements in geometry. In the previous chapter we used Szemerédi's Regularity Lemma to show Theorem 4.7 and my first proof to Theorem 4.6 is based on that lemma too. Here we give an example for another sort of application. In Chapter 4 we had results about large crossing or non-crossing subsets of line segments. The problem is more difficult if we are looking for large subset of segments such that they are *pairwise* crossing or *pairwise* non-crossing. The situation is even more complicated if we replace segments by other geometric objects. In most of the cases the intersection graph of certain geometric objects cannot contain some induced subgraphs. We have already seen examples in the previous chapter (Fig.4.1). As we see in this chapter, one can use this simple observation to say something about the clique or independence number of the intersection graph. With other words if you have an n-element set of geometric objects then a large subset ($>> \log n$) has the property that its elements are pairwise disjoint or pairwise crossing.

Given a graph G with vertex set $V(G)$ and edge set $E(G)$, let $\alpha(G)$ and $\omega(G)$ denote the size of the largest independent set (empty subgraph) and the size of the largest clique (complete subgraph) in G, respectively. A subset $U \subseteq V(G)$ is called *homogeneous*, if it is either an independent set or a clique. Denote $\hom(G)$ the size of the largest homogeneous set in G, i.e., let

$$\hom(G) = \max\left(\alpha(G), \omega(G)\right).$$

If H is not an induced subgraph of G, then we say that G is an H*-free* graph.

According to classical Ramsey theory, $\hom(G) \geq \frac{1}{2}\log_2 n$ for every graph G with n vertices [35], and there exists some G with $\hom(G) < 2\log_2 n$ (see

[27]). Erdős and Hajnal [32] raised the possibility that "the following could be true."

Conjecture 5.1 *For every graph H, there exists a positive $\epsilon = \epsilon(H)$ such that every H-free graph with n vertices has a homogeneous set whose size is at least n^ϵ.*

Erdős and Hajnal confirmed their conjecture for every graph H which belongs to the class $\mathcal{H}$ defined recursively as follows:

1. K_1, the graph consisting of a single vertex, belongs to $\mathcal{H}$;
2. if H_1 and H_2 are two vertex-disjoint graphs belonging to $\mathcal{H}$, then their disjoint union as well as the graph obtained from this union by connecting every vertex of H_1 to every vertex of H_2 belongs to $\mathcal{H}$.

Gyárfás [44] noticed that it follows from a well known result of Seinsche [77] that Conjecture 5.1 is also true for all graphs generated by the above rules starting with P_4, a simple path with 4 vertices, and K_1.

If Conjecture 5.1 is true for some graph H, then we say that H has the *Erdős-Hajnal property*.

Our first theorem extends both of these results.

For any graph H with vertex set $V(H) = \{v_1, \ldots, v_k\}$ and for any other graphs, $F_1, \ldots, F_k$, let $H(F_1, \ldots, F_k)$ be the graph obtained from H by replacing each v_i with a copy of F_i, and joining a vertex of the copy of F_i to a vertex of a copy of F_j, $j \neq i$, if and only if $v_i v_j \in E(H)$. The copies of F_i, $i = 1, \ldots, k$, are supposed to be vertex disjoint.

Theorem 5.1 *If $H, F_1, \ldots, F_k$ have the Erdős-Hajnal property, then so does $H(F_1, \ldots, F_k)$.*

In other words, the Erdős-Hajnal property is preserved by replacement. This enables us to verify that Conjecture 5.1 is true, e.g., for the graphs depicted in Figure 5.1, which answers some questions of Gyárfás [44].

No non-perfect graph is known to have the Erdős-Hajnal property. Unfortunately, in this respect Theorem 5.1 cannot offer any help. Indeed, according to a result of Lovász [60], which played a key role in his proof of the Weak

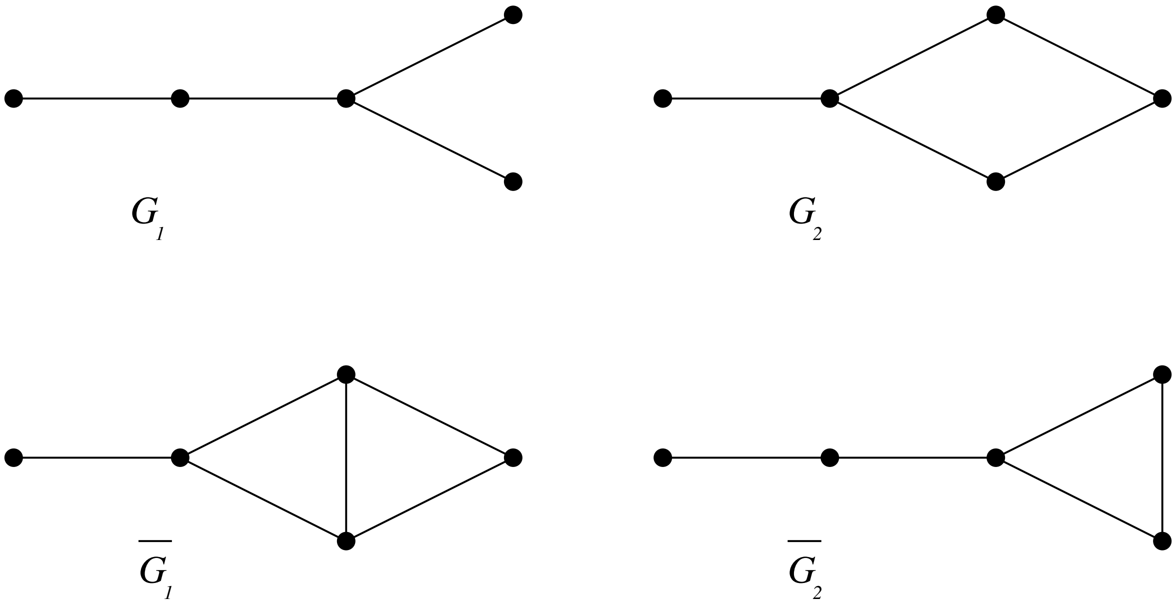

Figure 5.1: *G_1 and G_2 have the Erdős-Hajnal property*

Perfect Graph Conjecture [59], perfectness is also preserved by replacement. It is an outstanding open problem to decide whether the smallest non-perfect graph, the cycle of length 5, has the Erdős-Hajnal property or not. As Lovász pointed out, there is an even simpler unsolved problem.

Problem 5.1 *Does there exist a positive constant ϵ so that, for every graph G on n vertices such that neither G nor its complement $\bar{G}$ contains an induced odd cycle whose length is at least 5, we have $\hom(G) \geq n^\epsilon$?*

It is easy to formulate analogous questions for tournaments. A tournament with no directed cycle is called *transitive*. If a tournament has no subtournament isomorphic to T, then it is called T*-free*.

It is well known [24, 82] that every tournament of n vertices contains a transitive subtournament whose size is at least $c \log n$, and that this result is tight apart from the value of the constant.

Conjecture 5.2 *For every tournament T, there exists a positive $\epsilon = \epsilon(T)$ such that every T-free tournament with n vertices has a transitive subtournament whose size is at least n^ϵ.*

Theorem 5.2 *Conjecture 5.1 and Conjecture 5.2 are equivalent.*

In order to prove Theorem 5.2, we need a Ramsey-theoretic result for tournaments, which is interesting on its own right. A tournament T with a linear order $<$ on its vertex set is called an *ordered tournament* and is denoted by $(T, <)$. An ordered tournament $(T, <)$ is said to be a *subtournament* of another ordered tournament, $(T', <')$, if there is a function $f : V(T) \to V(T')$ satisfying the conditions

(i) $f(u) <' f(v)$ if and only if $u < v$,

(ii) $\overrightarrow{f(u)f(v)} \in E(T')$ if and only if $\vec{uv} \in E(T)$.

Theorem 5.3 *For any ordered tournament $(T, <)$, there exists a tournament T' such that, for every ordering $<'$ of T', $(T, <)$ is a subtournament of $(T', <')$. Moreover, if T has n vertices, there exists a T' with the required property with $O\left(n^3 \log^2 n\right)$ vertices.*

We further show that the $O(n^3 \log^2 n)$ estimate is not very far from being tight. In fact, if $(T, <)$ is any tournament on n vertices and T' satisfies the condition above for T, then T' must have at least $\Omega(n^2)$ vertices. The proof of the above theorem is very similar to the proof of the main result of [75], which deals with a similar statement for ordered induced subgraphs. This can be extended to hypergraphs as well.

By choosing a bigger tournament T', one can ensure a single tournament that contains all ordered tournaments on n vertices, in any ordering. Specifically, we prove the following.

Theorem 5.4 *There is a general bound $B > 0$ such that the following is true. Given an integer $N > B$, let n_0 be the largest integer such that*

$$\binom{N}{n_0} 2^{-\binom{n_0}{2}} \geq 1,$$

and put $n = n_0 - 2$. Then there exists an ordered tournament T' on N vertices such that in any ordering it contains every ordered tournament on n vertices.

Note that the above estimate for n is clearly tight, up to an *additive* error of 2. A similar statement holds for induced subgraphs, as shown in [13].

5.2 Graphs with the Erdős-Hajnal property

In this section we prove Theorem 5.1. Obviously, it is sufficient to show the following weaker version of the theorem.

Theorem 5.5 *Let H and F be graphs having the Erdős-Hajnal property, $V(H) = \{v_1, v_2, \ldots, v_k\}$. Then the graph $H(F, v_2, \ldots, v_k)$, obtained by replacing v_1 with F, also has this property.*

PROOF. Let H_0 denote the graph obtained from H by the deletion of v_1. For simplicity, write $H(F)$ for $H(F, v_2, \ldots, v_k)$. Let G be an $H(F)$-free graph with n vertices, and assume that $\hom(G) < n^{\epsilon(H)\delta}$. We would like to get a contradiction, provided that $\delta > 0$ is sufficiently small.

Let $m := \lceil n^\delta \rceil > k$. By the definition of $\epsilon(H)$, any m-element subset of $U \subset V(G)$ must induce at least one subgraph isomorphic to H. Otherwise, we would find a homogeneous subset of $m^{\epsilon(H)} > \hom(G)$ in the subgraph of G induced by U, which is impossible. Therefore, G has at least $\binom{n}{m}/\binom{n-k}{m-k}$ induced subgraphs isomorphic to H. For each of these subgraphs, fix an isomorphic embedding of H into G.

Since the number of embeddings of H_0 into G is smaller than $n(n-1)\cdots(n-k+2)$, there exists an embedding, which can be extended to an embedding of H in at least

$$M := \frac{\binom{n}{m}}{\binom{n-k}{m-k} n(n-1)\cdots(n-k+2)} \tag{5.1}$$

different ways. In other words, there are $k-1$ vertices, $v'_2, \ldots, v'_k \in V(G)$, and there exists an at least M-element subset $W \subset V(G)$ such that, for every $w \in W$,

$$f(v_1) = w, \;\; f(v_i) = v'_i \; (i = 2, \ldots, k)$$

is an isomorphic embedding of H into G.

Consider now the subgraph $G|_W$ of G induced by W. This graph must be F-free, otherwise G would not be $H(F)$-free. Since F has the Erdős-Hajnal property, we know that

$$\hom(G|_W) \geq |W|^{\epsilon(F)} \geq M^{\epsilon(F)}.$$

On the other hand,

$$n^{\epsilon(H)\delta} > \hom(G) \geq \hom(G|_W).$$

Comparing the last two inequalities and plugging in the value (5.1) for M, we obtain that

$$n^{\delta\epsilon(H)/\epsilon(F)} > \frac{\binom{n}{m}}{\binom{n-k}{m-k} n(n-1)\cdots(n-k+2)} =$$

$$\frac{n-k+1}{m(m-1)\cdots(m-k+1)} > n^{1-k\delta},$$

which gives the desired contradiction, provided that

$$\delta < \frac{\epsilon(F)}{\epsilon(H) + k\epsilon(F)}.$$

5.3 Ramsey-type theorems for tournaments

The proof of Theorem 5.3 uses the probabilistic method. The basic idea is a slightly simplified version of the main argument of Rödl and Winkler in [75]. We need the following lemma.(Here ij means multiplication between i and j).

Lemma 5.1 *Let $t > n > 1$ be two positive integers, and let $S = \{1, 2, \ldots, tn\}$ and let $g : S \to R = \{1, 2, \ldots, t\}$ be a function such that for every $p \in R$, we have $|\{i : g(i) = p\}| = n$. Further, let $f : S \to N = \{1, 2, \ldots, n\}$ be a random function obtained by choosing, for each element $i \in S$, randomly, independently, and with uniform distribution a value $f(i) \in N$. Let E be denote the event that there exist $i1, i2, \ldots, in$ such that $g(ij) \neq g(ik)$ for all $1 \leq j < k \leq n$, and $f(ij) = j$ for all $1 \leq j \leq n$.*

Then the probability that E does not hold is at most

$$\sum_{q=0}^{n-1} \binom{tn}{q} \frac{n^{q(n-1)}(n-1)^{tn-nq}}{n^{tn}} \leq \left(\frac{4et}{n}\right)^n e^{-t}.$$

PROOF. To estimate the number of functions f for which the event E fails, we argue as follows. Given such an f, let i_1 be the smallest integer (if it exists) such that $f(i1) = 1$. Assuming $i1, i2, \ldots, i(j-1)$ have already been defined, and assuming that $f(is) = s$ for all $s < j$ and that the elements $g(is), s < j$, are pairwise distinct, let ij be the smallest integer (if it exists) satisfying $f(ij) = j$ and $g(ij) \neq g(is)$ for all $s < j$. Note that, since the event E fails, this process must terminate after some $q \leq n-1$ elements is have been defined. Note also that if k is an integer satisfying $i(s-1) < k < is$, and $g(k)$ differs from $g(ij)$ for all $j \leq q$ (or even just for all $j < s$), then $f(k)$ cannot be equal to s (since otherwise we would have defined $is = k$). Since there is a similar restriction for the value of $f(k)$ for $k < i1$ and for $k > iq$, it follows that once the sequence $i1, i2 \ldots, iq$ has been defined, the function $f(k)$ can attain at most $n-1$ values for all but at most $tn - nq$ elements for k. Therefore, the total number of functions f for which the event E fails is at most

$$\sum_{q=0}^{n-1} \binom{tn}{q} n^{q(n-1)} (n-1)^{tn-nq}.$$

Since the total number of possible functions f is n^{tn}, the probability that E does not hold is at most

$$\sum_{q=0}^{n-1} \binom{tn}{q} \frac{n^{q(n-1)}(n-1)^{tn-nq}}{n^{tn}} \leq$$

$$\sum_{q=0}^{n-1} \left(\frac{etn}{q}\right)^q \left(\frac{n^n}{(n-1)^n}\right)^q \frac{1}{n^q} \left(1 - \frac{1}{n}\right)^{tn}$$

$$\leq \left(\frac{et}{n}(1+\frac{1}{n-1})^n\right)^n e^{-t} \leq \left(\frac{4et}{n}\right)^n e^{-t}.$$

PROOF. (of Theorem 5.3) Let $(T, <)$ be an ordered tournament on the set $N = \{1, 2, \ldots, n\}$ of n vertices, ordered naturally. We may and will assume that n is sufficiently large. Let $c > 3$ be an absolute constant, and let t be the smallest integer satisfying $t > cn \log n$ such that $t-1$ is a prime. By the known estimates for the distribution of primes, $t = (1 + o(1))cn \log n$. Let P be a projective plane of order $t-1$. Each line of P contains precisely t points, and the number of points in P is $(t-1)^2 + t < t^2$. Replace each point $p \in P$ by a set S_p of n points, where all sets S_p are pairwise disjoint. Construct a tournament T' on the set $\cup_{p \in P} S_p$ of less than nt^2 vertices as follows. For every line l in P, let $f_l : \cup_{p \in l} S_p \to N = \{1, 2, \ldots, n\}$ be a random function, where each image $f_l(u)$ is chosen randomly, uniformly and independently in N, and the functions corresponding to different lines are chosen independently. For $u, v \in \cup_{p \in l} S_p$, where $u \in S_p$, $v \in S_{p'}$ and $p \neq p'$, let $\vec{uv}$ be a directed edge if and only if $\overrightarrow{f_l(u) f_l(v)}$ is a directed edge of T. If $f_l(u) = f_l(v)$ then uv can be directed arbitrarily, the edges with two endpoints in the same set S_p are oriented arbitrarily.

To complete the proof, we show that almost surely (that is, with probability tending to 1 as n tends to infinity), T' contains an ordered copy of T in any ordering. Fix an ordering $<'$ of T', and let us estimate the probability that in this ordering $(T', <')$ contains no ordered copy of T. For each line l in the projective plane, the ordering $<'$ induces an ordering of the tn vertices $\cup_{p \in l} S_p$. Let $S = (a_1, a_2, \ldots, a_{tn})$ be this induced ordering. Define $g(a_i) = p$ if $a_i \in S_p$. Then, for every $p \in l$, $|\{i : g(a_i) = p\}| = |S_p| = n$. Observe now that, by Lemma 5.1, the probability that $(T, <)$ is not a subtournament of the ordered subgraph of $(T', <')$ consisting of all edges running between distinct groups S_p $(p \in l)$, is at most $(\frac{4et}{n})^n e^{-t}$. This follows from the fact that, if the event E in Lemma 3.1 holds for $f = f_l$, then $a_{i_1}, \ldots, a_{i_n}$ induce a copy of T, as required. Since the events for distinct lines are totally independent, the probability that $(T', <')$ contains no ordered copy of $(T, <)$ is at most

$$\left((\frac{4et}{n})^n e^{-t} \right)^{(t-1)^2+t} = e^{-(1+o(1))c^3 n^3 \log^3 n}.$$

The total number of orderings of T' is $(n((t-1)^2+t))! \leq e^{(1+o(1))3c^2 n^3 \log^3 n}$, and as $c > 3$, by our choice, the probability that T' fails to contain a copy of T in some ordering is $o(1)$, completing the proof.

We next show that the $O(n^3 \log^2 n)$ upper bound cannot be replaced by $o(n^2)$. We need the following well-known result.

Lemma 5.2 ([22, 3]) *The number $|Aut(T)|$ of automorphisms of any tournament T on n vertices does not exceed $3^{(n-1)/2}$.*

Theorem 5.6 *There exists an absolute constant $b \geq \frac{1}{\sqrt{3}e^2}$ with the following property. Let $(T, <)$ be an ordered tournament on n vertices, and suppose T' is another tournament such that for every ordering $<'$ of T', $(T, <)$ is an induced subtournament of T. Then T' has at least bn^2 vertices.*

PROOF. Let N be the number of vertices of T'. Then the total number of induced labeled (but not necessarily ordered) copies of T in T' is at most $\binom{N}{n}|Aut(T)|$, which, by Lemma 5.2, does not exceed $(\frac{eN}{n})^n 3^{n/2}$. It follows that the probability that for a random ordering $<'$ of T', at least one of these copies is ordered, is at most

$$(\frac{eN}{n})^n 3^{n/2} \frac{1}{n!} \leq (\frac{\sqrt{3}e^2 N}{n^2})^n.$$

If $N < n^2/(\sqrt{3}e^2)$, this number is less than 1, implying that there is an ordering $<'$ with no ordered copy of $(T,<)$. Thus, we have $N \geq n^2/(\sqrt{3}e^2)$, completing the proof.

The discussion for tournaments can be easily adapted to induced subgraphs of graphs. A simple undirected graph H with a linear order $<$ on its vertex set is called an *ordered graph* and is denoted by $(H,<)$. An ordered graph $(H,<)$ is said to be an *induced subgraph* of another one, $(H',<')$, if there is a function $f : V(H) \to V(H')$ such that, for any $u, v \in V(H)$,

(i) $f(u) <' f(v)$ if and only if $u < v$,

(ii) $f(u)f(v) \in E(H')$ if and only if $uv \in E(H)$.

The proof of Theorem 5.3 can be easily modified to deal with ordered graphs, giving the following result of Rödl and Winkler.

Theorem 5.7 ([75]) *For any ordered graph $(H,<)$, there exists a graph H' such that, for every ordering $<'$ of H', $(H,<)$ is an induced subgraph of $(H',<')$. Moreover, if H has n vertices, there exists an H' with the required property with $O\left(n^3 \log^2 n\right)$ vertices.*

Note that there is no nontrivial analogue of Theorem 5.6, since the number of automorphisms of an undirected graph on n vertices can be as large as $n!$. In fact, if $(H,<)$ is an ordered complete graph on n vertices, then the graph $H' = H$ has only n vertices and contains an induced ordered copy of $(H,<)$ in any ordering.

Combining the above arguments with some known results about packing, we can extend the last result to induced hypergraphs as well. Moreover, the estimate for hypergraphs with no edge of size less than 3 is slightly better than the corresponding result for graphs.

A hypergraph H with a linear order $<$ on its vertex set is called an *ordered hypergraph* and is denoted by $(H,<)$. An ordered hypergraph $(H,<)$ is said to be an *induced subhypergraph* of another one, $(H',<')$, if there is a function $f : V(H) \to V(H')$ such that, for any $u, v \in V(H)$, $f(u) <' f(v)$ if and only if $u < v$, and a set of vertices forms an edge iff its image under f forms an edge.

Theorem 5.8 *For any ordered hypergraph $(H,<)$ in which each edge contains at least 3 vertices, there exists a hypergraph H' such that, for every ordering $<'$ of H', $(H,<)$ is an induced subhypergraph of $(H',<')$. Moreover, if H has n vertices, there exists an H' with the required property with $O\left(n^3\right)$ vertices.*

PROOF. Let $(H, <)$ be an ordered hypergraph on the set $N = \{1, 2, \ldots, n\}$ of n vertices, ordered naturally, where each edge of H is of size at least 3. Let c be an absolute constant such that $4ece^{-c} < 1/2$ ($c = 5$, for example, will do). Let t be the smallest prime satisfying $t > cn$ (then $t = (1 + o(1))cn$.) As described in [54], there is a simple, explicit construction of a family L of t^3 subsets of a set P of size t^2 such that each member of L is of cardinality t and the intersection of no two members of L is of size more than 2. Replace each element $p \in P$ by a set S_p of n points, where all sets S_p are pairwise disjoint. Construct a hypergraph H' on the set $\cup_{p \in P} S_p$ of nt^2 vertices as follows. For every $l \in L$, let $f_l : \cup_{p \in l} S_p \to N = \{1, 2, \ldots, n\}$ be a random function, where each image $f_l(u)$ is chosen, randomly, uniformly and independently in N, and the functions corresponding to different members $l \in L$ are chosen independently. If $u_1, \ldots, u_r$ are vertices in $\cup_{p \in l} S_p$, then $\{u_1, u_2, \ldots, u_r\}$ is an edge of H' iff the vertices u_i belong to pairwise distinct sets S_p, and $\{f_l(u_1), f_l(u_2), \ldots, f_l(u_r)\}$ is an edge of H. Note that, since the intersection of any two distinct members of L is of size at most 2, and H has no edges with fewer than 3 vertices, none of the edges defined above can lie in the union $\cup_{p \in l'} S_p$, for any $l' \in L, l' \neq l$.

To complete the proof, we show that almost surely H' contains an ordered induced copy of H in any ordering. Fix an ordering $<'$ of H', and let us estimate the probability that in this ordering $(H', <')$ contains no ordered induced copy of H. For each $l \in L$, the ordering $<'$ induces an ordering of the nt vertices $\cup_{p \in l} S_p$. Let $S = (a_1, a_2, \ldots, a_{tn})$ be this induced ordering. Define $g(a_i) = p$ if $a_i \in S_p$. Then, for every $p \in l$, $|\{i : g(a_i) = p\}| = |S_p| = n$. Observe now that, by Lemma 5.1, the probability that $(H, <)$ is not an induced subhypergraph of the induced ordered subhypergraph of $(H', <')$ on S, is at most $(\frac{4et}{n})^n e^{-t} \leq 2^{-n}$. This is true, because if the event E in Lemma 3.1 holds for $f = f_l$, then the vertices $a_{i_1}, \ldots, a_{i_n}$ induce a copy of T, as required. Since the events for distinct sets $l \in L$ are totally independent, the probability that $(H', <')$ contains no ordered copy of $(H, <)$ is at most

$$(2^{-n})^{t^3} = 2^{-(1+o(1))c^3 n^4}.$$

The total number of orderings of H' is $(nt^2)! \leq e^{(1+o(1))3c^2 n^3 \log n}$, and thus the probability that H' fails to contain a copy of H in some ordering is $o(1)$, completing the proof.

It is worth noting that the argument in the proof of Theorem 5.6 also works for hypergraphs whose group of automorphisms is not too large. In particular, if the hypergraph H in the statement of the last theorem has no nontrivial automorphisms, then the number of vertices of any hypergraph H' satisfying the assertion of the theorem must be at least $\Omega(n^2)$.

Returning to tournaments, we now describe a proof of Theorem 5.4, using Talagrand's Inequality [86]. An alternative proof can be given using the methods of [13].

PROOF. (of Theorem 5.4) Let N, n_0 and n be as in the statement of the theorem, and let T' be a random tournament on the vertices $1, 2, \ldots, N$, obtained by choosing, for each pair of vertices i, j of T', randomly, uniformly, and independently, either the edge $\vec{ij}$ or the edge $\vec{ji}$. Whenever it is needed, we assume that N is sufficiently large. To complete the proof, we show that almost surely in every ordering, T' contains an ordered copy of every tournament on n vertices. To this end, fix an ordering $<'$ of T', and fix an ordered tournament T on n vertices. We use Talagrand's Inequality (see, e.g., [2], Chapter 7) to estimate the probability that in this ordering T' contains no ordered copy of T. The computation here is very similar to the one estimating the probability that the clique number of the random graph $G(n, 1/2)$ is less than its expected value by at least 2.

For each set K of n vertices of T', let B_K be the event that the induced subgraph of $(T', <')$ on K is an ordered copy of $(T, <)$. Then the probability $\Pr(B_K)$ of each event B_K is precisely $2^{-\binom{n}{2}}$. Define $\mu = \binom{N}{n} 2^{-\binom{n}{2}}$, and note that this is the expected number of ordered copies of $(T, <)$ in $(T', <')$. A simple computation shows that the number n_0 defined in the statement of the theorem satisfies $n_0 = (1 + o(1))2 \log_2 N$, implying that for the function $f(m) = \binom{N}{m} 2^{-\binom{m}{2}}$ and for every m close to n_0, we have $f(m + 1)/f(m) = N^{-1+o(1)}$. Therefore, $\mu \geq N^{2-o(1)}$.

For two subsets K and K', each containing n vertices of T', let $K \sim K'$ stand for the fact that $2 \leq |K \cap K'| \leq n-1$. Define, further, $\Delta = \sum_{K \sim K'} \Pr(B_K \wedge B_{K'})$, where the sum ranges over all ordered pairs (K, K') with $K \sim K'$. Therefore, $\frac{1}{2}\Delta$ is the expected number of pairs of ordered copies of T that share an edge.

The technical part of the proof is a careful estimate of the quantity Δ/μ^2. Observe that if $|K \cap K'| = i$ (≥ 2) then

$$\Pr(B_K \wedge B_{K'}) \leq 2^{-2\binom{n}{2}+\binom{i}{2}}.$$

In fact, $\Pr(B_K \wedge B_{K'})$ is equal either to zero, or to the right-hand side of the above expression. Thus, it follows that $\Delta \leq \sum_{i=2}^{n-1} \Delta_i$, where

$$\Delta_i = \binom{N}{n}\binom{n}{i}\binom{N-n}{n-i}2^{-2\binom{n}{2}+\binom{i}{2}}.$$

Therefore,

$$\frac{\Delta_i}{\mu^2} = \frac{\binom{N}{n}\binom{n}{i}\binom{N-n}{n-i}2^{-2\binom{n}{2}+\binom{i}{2}}}{\binom{N}{n}^2 2^{-2\binom{n}{2}}}$$

$$= \frac{\binom{n}{i}\binom{N-n}{n-i}2^{\binom{i}{2}}}{\binom{N}{n}} \leq (\frac{n}{N})^i \binom{n}{i} 2^{\binom{i}{2}} \leq \left(\frac{n^2}{N}2^{(i-1)/2}\right)^i.$$

It follows that

$$\frac{\Delta_2}{\mu^2} \leq 2\frac{n^4}{N^2}, \tag{5.2}$$

and that for each i satisfying, say, $3 \leq i \leq 100$, we have

$$\frac{\Delta_i}{\mu^2} = O(\frac{n^6}{N^3}). \tag{5.3}$$

Furthermore, for $100 < i \leq 1.9\log_2 N$, we have

$$\frac{\Delta_i}{\mu^2} \leq (\frac{n^2}{N^{0.05}})^i < \frac{1}{N^5}. \tag{5.4}$$

For every i in the range $1.9\log_2 N \leq i \leq n-1$, put $i = n-j$, and note that $1 \leq j \leq (0.1+o(1))\log_2 N$ and

$$\frac{\Delta_i}{\mu^2} \leq \frac{1}{N^{2-o(1)}}\frac{\Delta_i}{\mu} = \frac{1}{N^{2-o(1)}}\frac{\binom{N}{n}\binom{n}{j}\binom{N-n}{j}2^{-\binom{n}{2}-j(n-j)-\binom{j}{2}}}{\binom{N}{n}2^{-\binom{n}{2}}}$$

$$\leq \frac{1}{N^{2-o(1)}}\binom{n}{j}\binom{N-n}{j}2^{-j(n-j)} \leq \frac{1}{N^{2-o(1)}}\left(nN2^{-(n-j)}\right)^j$$

$$\leq \frac{1}{N^{2-o(1)}}\left(\frac{n}{N^{0.9-o(1)}}\right)^j \leq \frac{1}{N^{2.9-o(1)}}.$$

Combining the last inequality with inequalities (5.2), (5.3) and (5.4), we conclude that $\frac{\Delta}{\mu^2} \leq (2+o(1))\frac{n^4}{N^2}$.

Let $X = h(T')$ denote the maximum number of pairwise edge-disjoint ordered copies of T in $(T', <')$. We claim that the expected value of $X = h(T')$ satisfies

$$E(X) \geq (\frac{1}{4} + o(1))\frac{N^2}{n^4}. \tag{5.5}$$

To see this, define $p = \frac{N^2}{2n^4\mu}$, and note that, by a simple computation, we have $p < 1$. Let $\mathcal{S}$ be a random collection of ordered copies of T in T' obtained by choosing each ordered copy of T in T' to be a member of $\mathcal{S}$, randomly and independently, with probability p. The expected number of copies of T in $\mathcal{S}$ is $p\mu$, and the expected number of pairs of members of $\mathcal{S}$ that share an edge is $\frac{1}{2}\Delta p^2$. By omitting an arbitrarily chosen member of each such pair, we obtain a collection of pairwise edge-disjoint copies of T whose expected number is at least $p\mu - \frac{1}{2}\Delta p^2$. Thus

$$E(X) \geq p\mu - \frac{p^2\Delta}{2} \geq \frac{N^2}{2n^4} - \frac{N^4}{8n^8}(2 + o(1))\frac{n^4}{N^2} = (\frac{1}{4} + o(1))\frac{N^2}{n^4},$$

establishing (5.5).

To apply Talagrand's Inequality (in the form presented, for example, in [2], Chapter 7), note that $h(T')$ is a Lipschitz function, that is $|h(T') - h(T'')| \leq 1$ if T', T'' differ in the orientation of at most one edge. Note also that h is f-certifiable for $f(s) = \binom{n}{2}s$. That is, whenever $h(T') \geq s$ there is a set of at most $\binom{n}{2}s$ oriented edges of T' such that for every ordered tournament T'' which agrees with T' on these edges, we have $h(T'') \geq s$.

By Talagrand's Inequality we conclude that for every b and t

$$Pr[X \leq b - t\sqrt{f(b)}]\ Pr[X \geq b] \leq e^{-t^2/4}. \tag{5.6}$$

Let B denote the median of $X = h(T')$. Without trying to optimize the absolute constants, we claim that

$$B \geq \frac{N^2}{16n^4}. \tag{5.7}$$

Indeed, assume this is false, and apply (5.6) with $b = \frac{N^2}{8n^4}$ and $t = \frac{N}{4n^3}$. As $f(b) = \binom{n}{2}\frac{N^2}{8n^4} \leq \frac{N^2}{16n^2}$, we obtain that

$$Pr[X \leq \frac{N^2}{8n^4} - \frac{N^2}{16n^4}]\ Pr[X \geq \frac{N^2}{8n^4}] \leq e^{-N^2/(64n^6)}.$$

Since, by assumption, $B < \frac{N^2}{16n^4}$, the first term of the left-hand side is at least $1/2$, and we conclude that

$$Pr[X \geq \frac{N^2}{8n^4}] \leq 2e^{-N^2/(64n^6)}.$$

As $X = h(T') \leq \binom{N}{n}$ for every T', this implies that

$$E(X) \leq \frac{N^2}{8n^4} + \binom{N}{n} 2e^{-N^2/(64n^6)} = \frac{N^2}{8n^4} + o(1),$$

contradicting (5.5) and hence proving (5.7).

We can now apply (5.6) with $b = \frac{N^2}{16n^4}$ and $t = \frac{N}{4n^3}$ to obtain that

$$Pr[X = 0]Pr[X \geq b] \leq Pr[X \leq b - t\sqrt{f(b)}]Pr[X \geq b] \leq e^{-N^2/(64n^6)}.$$

By (5.7), we have $Pr[X \geq b] \geq 1/2$, and hence $Pr[X = 0] \leq 2e^{-N^2/(64n^6)}$. Thus, we have proved that, for every fixed ordering of T' and for every fixed ordered T, the probability that T' contains no ordered copy of T is at most $2e^{-N^2/(64n^6)}$. Since the total number of orderings of T' is less than $N^N = e^{N \log N}$ and the total number of tournaments T on n vertices is $2^{\binom{n}{2}}$ we conclude that the probability that $(T', <')$ fails to contain some tournament of size n in some ordering is at most

$$e^{N \log N} 2^{\binom{n}{2}} 2e^{-N^2/(64n^6)} = o(1).$$

This completes the proof.

The above proof can be modified to deal with graphs in the place of tournaments. We obtain the following, which is a very slight numerical improvement of the main result in [13].

Theorem 5.9 (see also [13]) *Given an integer N, let n_0 be the largest integer such that*

$$\binom{N}{n_0} 2^{-\binom{n_0}{2}} \geq 1,$$

and put $n = n_0 - 2$. Then, for all sufficiently large N, the following holds almost surely. The random graph $G(N, 1/2)$ contains, in any ordering, an induced copy of every ordered graph on at most n vertices.

5.4 Tournaments and H-free graphs

In this section, we prove Theorem 5.2. We need the following well-known

Lemma 5.3 ([35]) *For any two total orderings of the same (k^2+1)-element set V, there is a $(k+1)$-element subset $U \subseteq V$ such that either the order of any two elements of U is the same, or the order of any two elements is opposite in the two orderings.*

We say that a tournament T has the *Erdős-Hajnal property* if there exists a positive $\epsilon = \epsilon(T)$ such that every T-free tournament with n vertices has a transitive subtournament whose size is at least n^ϵ.

To any tournament T and to any ordering $<$ of its vertex set, assign an ordered graph $(H(T), <)$ on the same vertex set, as follows. Join two vertices $u < v$ by an edge of $H(T)$ if and only if the edge connecting them in T was directed towards v. Similarly, assign to any ordered graph $(H, <)$ an ordered tournament $(T(H), <)$ with the same vertex set, by connecting $u < v$ with an edge directed towards v if $uv \in E(H)$ and with an edge directed towards u if $uv \notin E(H)$.

Now we have everything needed for the

PROOF. (of Theorem 5.2) Assume first that Conjecture 5.1 is true, i.e., every graph has the Erdős-Hajnal property. Let T be a tournament. We want to show that T also has the Erdős-Hajnal property.

Choose an arbitrary ordering $<$ of the vertex set of T. Applying Theorem 5.7 to the ordered graph $(H(T), <)$ associated with T and $<$, we obtain that there exists a graph H' with the property that, for any ordering $<'$ of H', $(H(T), <)$ is an induced subgraph of $(H', <')$. By Conjecture 5.1, there exists an $\epsilon(H') > 0$ such that every H'-free graph with n vertices has a homogeneous subset of size at least $n^{\epsilon(H')}$.

Consider now a T-free tournament T' with n vertices and an ordering $<'$ of $V(T')$. Then the ordered graph $(H(T'), <')$ associated with them cannot contain an induced subgraph isomorphic to H' (because, no matter how it is ordered, this would yield a copy of T in T'). Thus, $H(T')$ must have a homogeneous set of size at least $n^{\epsilon(H')}$. However, a homogeneous set in $(H(T'), <')$ corresponds to a transitive subtournament in T'.

The proof of the reverse statement is very similar, but the roles of graphs and tournaments have to be switched. Assume that Conjecture 5.2 is true, and let H be an arbitrary graph. To establish that H has the Erdős-Hajnal property, fix a linear order $<$ on $V(H)$, and denote the associated ordered tournament by $(T(H), <)$.

By Theorem 5.3, there exists a tournament T' with the property that, for any ordering $<'$ of T', $(T(H), <)$ is a subtournament of $(T', <')$. By Conjecture 5.2, there exists an $\epsilon(T') > 0$ such that every T'-free tournament with n vertices has a transitive subtournament of size at least $n^{\epsilon(T')}$.

Consider now an H-free graph H' with n vertices and an ordering $<'$ of $V(H')$. Then the ordered tournament $(T(H'), <')$ associated with them cannot contain a subtournament isomorphic to T' (because, no matter how it is ordered, this would yield a copy of H in H'). Thus, $T(H')$ must have a transitive subtournament of size at least $n^{\epsilon(H')}$. However, by Lemma 5.3, any such subtournament has at least $n^{\epsilon(H')/2}$ vertices such that, with respect to the ordering $<'$, either all edges connecting them are directed towards their larger endpoints, or all of them are directed towards their smaller endpoints. These vertices induce a complete or an empty subgraph of H', respectively.

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