# Order Polytope

Let $G = (V, E)$ be an acyclic digraph with $V = [n] := \{1, 2, \ldots, n\}$. One might want to consider $G$ as a representation of the partially ordered set (poset) $V$: $i > j$ if and only if there is a directed path from node $i$ to node $j$.

A permutation $\pi$ of $[n]$ is called a *linear extension* of $G$ (or the associated poset $V$) if $\pi^{-1}(i) > \pi^{-1}(j)$ for every edge $\overrightarrow{ij} \in E$.

There are several fundamental problems associated with linear extensions of a partially ordered set. Computing the number $\#_{\mathrm{LE}}(G)$ of linear extensions is known to be hard, #P-complete by Brightwell and Winkler [3], that was conjectured by Linial [10].

There are polynomial algorithms to list all linear extensions, see e.g. [9, 2, 11]. Here "polynomial" means the computational time is bounded by a polynomial function of input and output sizes.

Khachiyan [8] showed that the hardness of the counting problem implies computing the volume of an H-polytope is #P-hard. This reduction uses the following polytope associated with a poset.

Let $P_{\mathrm{LE}}(G)$ be the polytope in $R^n$ defined by

$$
\begin{aligned}
P_{\mathrm{LE}}(G) = \{x \in R^n \mid 1 \geq x_i \geq 0 & \text{ for all i=1,2,...,n, and} \\
& x_i \geq x_j \text{ for all directed edges } \overrightarrow{ij} \in E\}.
\end{aligned}
$$

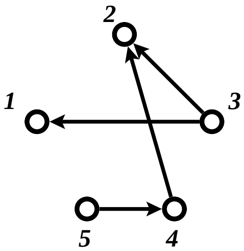
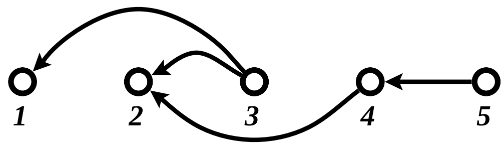
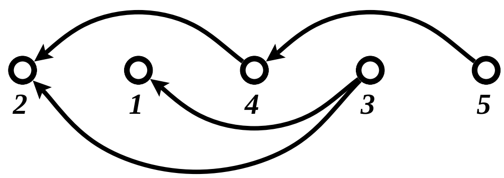
The polytope $P_{\mathrm{LE}}(G)$ is known as the *order polytope* of $G$ (of the associated partial order). The following properties are known [12, 10].

**(a)** $P_{\mathrm{LE}}(G)$ is a 0/1-polytope, i.e. all extreme points are in $\{0, 1\}^n$.

**(b)** There is a one-to-one correspondence between the extreme points and the ideals of the poset. Here, an *ideal* (sometimes called "upper ideal" or "up-set") of a poset $V$ is a subset $S$ of $V$ such that if $i, j \in V$, $j > i$ and $i \in S$ then $j \in S$.

**(c)** The volume of $P_{\mathrm{LE}}(G)$ is equal to $\#_{\mathrm{LE}}(G)/n!$.

There are several combinatorial optimization problems related to linear extensions of acyclic graphs. For example, the minimum feedback arcset problem [7] and the strongly connected reorientation problem [4] can be stated as geometric problems on closely related polyhedra and arrangements, see [6].

| **Input**<br>$G$ | **Listing**<br>$\lambda_{\mathrm{LE}}(G)$ | **Counting**<br>$\#_{\mathrm{LE}}(G)$ |
|---|---|---|
| <br><br>acyclic digraph | <br><br>$\pi_0$= 12345<br><br><br>$\pi_1$= 21435<br>$\pi_2 = 21345$<br>$\pi_3 = 12435$<br>$\pi_4 = 24135$<br>$\pi_5 = 12453$<br>$\pi_6 = 21453$<br>$\pi_7 = 24153$<br>$\pi_8 = 24513$ | 9 |

## Results of computation with polyhedral computation codes

*Polyhedra format*, defined by Avis and Fukuda, provides a simple standard way to write both H- and V-representation of a general polyhedron. An inequality system of form $b + Ax \geq 0$ defines an H-polyhedron and its Polyhedra format is

```
H-representation
begin
  m  d+1  <number type>
  b   A
end
```

where $A$ is an $m \times d$ matrix and <number type> must be one of integer, rational or real.

If all generators of a polyhedron is known, then it is a V-polyhedron. For example, if $P$ is a V-polyhedron given by $n$ generating points and $s$ generating directions (rays) as $P = conv(v_1, \ldots, v_n) + nonneg(r_1, \ldots, r_s)$. Then its Polyhedra V-format for $P$ is

```
V-representation
begin
 n+s  d+1  <numbertype>
 1   v1
  .
 1   vn
 0   r1
  .
 0   rs
end
```

The linear extension polytope in the example above is an H-polytope and it can be written in polyhedra format as

```
H-representation
begin
  14  6  integer
  0 -1  0  1  0  0
  0  0 -1  1  0  0
  0  0 -1  0  1  0
  0  0  0  0 -1  1
  0  1  0  0  0  0
  0  0  1  0  0  0
  0  0  0  1  0  0
  0  0  0  0  1  0
  0  0  0  0  0  1
  1 -1  0  0  0  0
  1  0 -1  0  0  0
  1  0  0 -1  0  0
  1  0  0  0 -1  0
  1  0  0  0  0 -1
end
```

For example, cdd, cdd+ and scdd (sample program in cddlib, Version 0.94h) [5] takes the above file as input and outputs a V-representation of the polytope.

```
V-representation
begin
 11 6 rational
 1 0 0 0 0 0
 1 0 0 1 0 0
 1 1 0 1 0 0
 1 0 0 0 0 1
 1 0 0 1 0 1
 1 1 0 1 0 1
 1 0 0 0 1 1
 1 0 0 1 1 1
 1 1 0 1 1 1
```

```
 1 0 1 1 1 1
 1 1 1 1 1 1
end
```

The polytope has eleven extreme points all of which are 0/1, as these polytopes are 0/1 in general. (The first 1 in each row indicates that the rest is an extreme point. For an unbounded polyhedron, each extreme ray generator is represented by a row starting with 0.)

Avis' lrs [1] does the same transformation by a completely different algorithm. Futhermore, lrs computes the volume of the polytope if it is given by V-representation. For example, it outputs the value 3/40 for the order polytope above, which is 9/5!.

# References

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