Cyclic Polytope

The convex hull of \( n \) distinct points on the moment curve \( \{m(t) = (t^1, t^2, \ldots, t^d)^T : t \in R \} \) in \( R^d \) is known as a cyclic polytope. It is known that its combinatorial structure (i.e. its face lattice) is uniquely determined by \( n \) and \( d \). Thus we often write \( c(d,n) \) to denote any polytope combinatorially equivalent to a cyclic \( d \)-polytope with \( n \) vertices.

Let \( f_k(P) \) denote the number of \( k \)-faces in a polytope \( P \). McMullen’s Upper Bound Theorem shows that the maximum of \( f_k(P) \) over all \( d \)-polytopes \( P \) with \( n \) vertices is attained by the cyclic polytopes for all values of \( k = 1, 2, \ldots, d-1 \).

**Theorem 1 (Upper Bound Theorem)** For any \( d \)-polytope with \( n \) vertices,

\[
f_k(P) \leq f_k(c(d,n)), \quad \forall k = 1, \ldots, d-1,
\]

holds.

The number of \( k \)-faces of a cyclic polytope \( c(d,n) \) can be explicitly given and thus one can evaluate the order of the upper bound in terms of \( n \) and \( d \).

**Theorem 2** For \( d \geq 2 \) and \( 0 \leq k \leq d-1 \),

\[
f_k(c(d,n)) = \sum_{r=0}^{[d/2]} \binom{r}{d-k-1} \binom{n-d+r-1}{r} + \sum_{r=[d/2]+1}^{d} \binom{r}{d-k-1} \binom{n-r-1}{d-r}.
\]

In particular, by using the binomial identity

\[
\binom{p+q+1}{q} = \binom{p+q}{q} + \binom{p+q-1}{q-1} + \cdots + \binom{p+1}{1} + \binom{p}{0},
\]

we have

\[
f_{d-1}(c(d,n)) = \binom{n-\lfloor d/2 \rfloor}{n-d} + \binom{n-\lfloor d/2 \rfloor - 1}{n-d} = \binom{n-\lceil d/2 \rceil}{\lfloor d/2 \rfloor} + \binom{n-\lceil d/2 \rceil - 1}{\lfloor d/2 \rfloor - 1} = O(n^{\lfloor d/2 \rfloor}) \quad \text{for any fixed } d.
\]

For example,

\[
\begin{array}{cccccc}
P & f_0 & f_1 & f_2 & f_3 & f_4 \\
c(5,10) & 10 & 45 & 100 & 105 & 42 \\
c(5,20) & 20 & 190 & 580 & 680 & 272 \\
c(5,30) & 30 & 435 & 1460 & 1755 & 702 \\
\end{array}
\]

The upper bound theorem can be written in dual form which gives, for example, the maximum number of vertices in a \( d \)-polytope with \( m \) facets.

**Theorem 3 (Upper Bound Theorem in Dual Form)** For any \( d \)-polytope with \( m \) facets,

\[
f_k(P) \leq f_{d-k-1}(c(d,m)), \quad \forall k = 0, 1, \ldots, d-2,
\]

holds.

The original proof of the Upper Bound Theorem is in [2,3]. There are different variations, see [1,4,5].
References


