TOPCOM: Triangulations of Point Configurations and Oriented Matroids - Jörg Rambau

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Basics

- \( A \subset \mathbb{R}^d \): \( d \)-dimensional configuration of \( n \) points
- \( k \)-simplex: sub-configuration of \( A \) consisting of \( k + 1 \) affinely independent points
- triangulation: collection \( T \) of \( d \)-simplices whose convex hulls cover \( \text{conv}(A) \) and intersect properly:
  \[ \forall \sigma, \tau \in T : \text{conv}(\sigma \cap \tau) = \text{conv}(\sigma) \cap \text{conv}(\tau) \]
Chirotepe

- Chirotepe: function \( (d+1) \choose A \rightarrow \{+, -, 0\} \) (gives the orientation of every \( d + 1 \)-subset of \( A \))

- Fact: Chirotepe contains all the information we need!
What can we do with TOPCOM?

Example:

\begin{tikzpicture}
  \coordinate (0) at (0,0);
  \coordinate (1) at (1,1);
  \coordinate (2) at (3,1);
  \coordinate (3) at (5,0);
  \coordinate (4) at (1,5);
  \coordinate (5) at (0,0);

  \draw (0) -- (1) -- (2) -- (3) -- (0);
  \draw (0) -- (4) -- (3);
  \draw (1) -- (4) -- (2);
  \draw (2) -- (4) -- (3);

  \node at (0) [above] {0};
  \node at (1) [above] {1};
  \node at (2) [above] {2};
  \node at (3) [above] {3};
  \node at (4) [above] {4};

  \node at (0) [left] {(0, 0)};
  \node at (1) [left] {(1, 1)};
  \node at (2) [left] {(3, 1)};
  \node at (3) [left] {(5, 0)};
  \node at (4) [left] {(1, 5)};
\end{tikzpicture}
Compute the chirotope

\[ \chi(i_1, i_2, \ldots, i_{d+1}) = \text{sign}(\det(a_{i_1}, a_{i_2}, \ldots, a_{i_{d+1}})) \]
Construct a Placing Triangulation

- $\mathcal{A}_k$: set of points that is already triangulated
- $T_k$: placing triangulation of $\mathcal{A}_k$
- $\mathcal{F}_k$: set of all boundary facets of $T_k$ that are interior in $\mathcal{A}$

Start with a $d$-simplex
In each step, add a point $a_{k+1}$ and all simplices $F \cup a_{k+1}$ for which $F \in \mathcal{F}_k$ is visible from $a_{k+1}$
Stop when $\mathcal{F}_k$ is empty
Remarks:

- The resulting triangulation depends on the numbering of the points.
- It's possible that not all points are used for the triangulation.
- To get a triangulation using all points (fine triangulation), the missing points are added one by one by 'flipping-in'.
Flips

Flip: exchange between the two possible triangulations in a subset of $d + 2$ points

$d = 1$:

$d = 2$:
Explore a flip-graph component

Flip graph: triangulations as vertices, two triangulations are connected if they differ by a flip
Explore a flip-graph component

The flip-graph of a regular hexagon
Check a potential triangulation

Theorem

Let $A$ be a full-dimensional point configuration in $\mathbb{R}^d$. A set $T$ of $d$-simplices of $A$ is a triangulation of $A$ if and only if

- For every pair $S$, $S'$ of simplices in $T$, there exists no circuit $(Z^+, Z^-)$ in $A$ with $Z^+ \subseteq S$ and $Z^- \subseteq S'$ (Intersection Property).

- For each facet of a simplex $S$ in $T$ there is another simplex $S' \neq S$ having $F$ as a facet, or $F$ is contained in a facet of $A$ (Union Property).

Circuit: partition of a minimal affinely dependent set into $Z^+$ and $Z^-$ such that $\text{conv}(Z^+) \cap \text{conv}(Z^-) \neq \emptyset$
References

- http://www.rambau.wm.uni-bayreuth.de/TOPCOM
- Francisco Santos, Triangulations of polytopes: personales.unican.es/santosf/Talks/icm2006.pdf