Matrix Calculus:

Given the following set of two equations in two unknowns:

$$\left| \begin{array}{l} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{array} \right| \qquad (1)$$

We wish to solve for x_1 and x_2.

$$\Rightarrow \quad \begin{array}{l|c} a_{22}a_{11}x_1 + a_{22}a_{12}x_2 = a_{22}b_1 & + \\ a_{12}a_{21}x_1 + a_{12}a_{22}x_2 = a_{12}b_2 & - \\ \hline \end{array}$$

$$(a_{11}a_{22} - a_{12}a_{21})\,x_1 = a_{22}b_1 - a_{12}b_2$$

$$\Rightarrow \quad x_1 = \frac{a_{22}b_1 - a_{12}b_2}{a_{11}a_{22} - a_{12}a_{21}}$$

$$\Rightarrow \quad \begin{array}{l|c} a_{21}a_{11}x_1 + a_{21}a_{12}x_2 = a_{21}b_1 & - \\ a_{11}a_{21}x_1 + a_{11}a_{22}x_2 = a_{11}b_2 & + \\ \hline \end{array}$$

$$(a_{11}a_{22} - a_{12}a_{21})\,x_2 = a_{11}b_2 - a_{21}b_1$$

$$\Rightarrow \quad x_2 = \frac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{12}a_{21}}$$

$$\Rightarrow \left| \begin{aligned} x_1 &= \frac{a_{22}}{a_{11}a_{22} - a_{12}a_{21}} \cdot b_1 + \frac{-a_{12}}{a_{11}a_{22} - a_{12}a_{21}} \cdot b_2 \\ x_2 &= \frac{-a_{21}}{a_{11}a_{22} - a_{12}a_{21}} \cdot b_1 + \frac{a_{11}}{a_{11}a_{22} - a_{12}a_{21}} \cdot b_2 \end{aligned} \right| \quad (2)$$

Let us introduce a shorthand notation for the original problem (1):

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

or:

$$A \cdot \underline{x} = \underline{b}$$

where: $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in \mathbb{R}^{2\times 2}$

is a square matrix of dimensions 2×2,

and:

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} ; \quad \underline{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} ; \quad \underline{x}, \underline{b} \in \mathbb{R}^2$$

are column vectors of length 2. The rules of product $A \cdot \underline{x}$ are implicitly defined by the meaning of this notation as a shorthand for problem (1).

The solution (2) has the same structure as (1), thus, we can write:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \dfrac{a_{22}}{a_{11}a_{22}-a_{12}a_{21}} & \dfrac{-a_{12}}{a_{11}a_{22}-a_{12}a_{21}} \\ \dfrac{-a_{21}}{a_{11}a_{22}-a_{12}a_{21}} & \dfrac{a_{11}}{a_{11}a_{22}-a_{12}a_{21}} \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

or:

$$\underline{x} = A^{-1} \cdot \underline{b}$$

where:

$$A^{-1} = \begin{bmatrix} \dfrac{a_{22}}{a_{11}a_{22}-a_{12}a_{21}} & \dfrac{-a_{12}}{a_{11}a_{22}-a_{12}a_{21}} \\ \dfrac{-a_{21}}{a_{11}a_{22}-a_{12}a_{21}} & \dfrac{a_{11}}{a_{11}a_{22}-a_{12}a_{21}} \end{bmatrix}$$

is the <u>inverse</u> of matrix A.

We notice that the four denominators are identical. We can pull them out, and write:

$$A^{-1} = \frac{\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}}{a_{11}a_{22} - a_{12}a_{21}} = \frac{A^{+}}{|A|}$$

<u>where</u>: $A^{+} = \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \in \mathbb{R}^{2\times 2}$

is the <u>adjugate</u> of matrix A,

and $|A| = a_{11}a_{22} - a_{12}a_{21} \in \mathbb{R}^{1}$ is the <u>determinant</u> of matrix A.

We now can compute solutions of 3 equations in 3 unknowns in the same manner, but I shall not do so. Instead, I shall give you the general solution in the matrix/vector shorthand notation.

Let $A \in \mathbb{R}^{n \times n}$ be a square matrix of dimensions $n \times n$.

Let a_{ij} denote the element of A that is located at the intersection of the i^{th} row with the j^{th} column.

Let $\{a_{ij}\}$ be the set of all elements a_{ij}, i.e.

$$A = \{a_{ij}\}$$

<u>Let</u> A_{ij} be a matrix of dimensions $(n-1) \times (n-1)$ that has the same elements as A, except that the i^{th} row and j^{th} column have been removed, e.g.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\Rightarrow A_{23} = \begin{bmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{bmatrix}$$

Using these definitions, we can compute the determinant of an $n \times n$ matrix A as the weighted sum of n subdeterminants of dimensions $(n-1) \times (n-1)$.

$$|A| = \sum_{i=1}^{n} (-1)^{i+k} \cdot a_{ik} \cdot |A_{ik}|$$

$$= \sum_{j=1}^{n} (-1)^{k+j} \cdot a_{kj} \cdot |A_{kj}|$$

where k can be any number between 1..n. It is useful to pick k such that there are as many ∅'s in the row or column that is used as the common index, e.g.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & \emptyset & 5 \\ 6 & \emptyset & 7 \end{bmatrix}$$

We choose k=2 together with the first of the two formulae, thus:

$$|A| = -2 \cdot \begin{vmatrix} 4 & 5 \\ 6 & 7 \end{vmatrix} + \emptyset \cdot \begin{vmatrix} 1 & 3 \\ 6 & 7 \end{vmatrix}$$

$$- \emptyset \cdot \begin{vmatrix} 1 & 3 \\ 4 & 5 \end{vmatrix}$$

$$= -2 \cdot (4 \cdot 7 - 5 \cdot 6) = -2 \cdot (-2) = \underline{\underline{4}}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & \emptyset & \emptyset \end{bmatrix}$$

We choose $k = 3$ together with the second of the two formulae, thus:

$$|A| = 7 \cdot \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} - \emptyset \cdot \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} + \emptyset \cdot \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}$$

$$= 7 \cdot \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} = 7 \cdot (2 \cdot 6 - 3 \cdot 5)$$

$$= 7 \cdot (-3) = \underline{\underline{-21}}$$

Using these same definitions, we can compute the elements of the adjugate matrix as follows:

$$A^{+} = \{ (-1)^{i+j} \cdot |A_{ji}| \}$$

i.e., each element of A^{+} needs one subdeterminant. Since A^{+} contains n^2 elements, the complete A^{+} requires n^2 subdeterminants to be computed. Notice the reversal of the indices!

Example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$\Rightarrow A^{+} = \begin{bmatrix} \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} \\ -\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} \\ \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} \end{bmatrix}$$

$$\Rightarrow A^{+} = \begin{bmatrix} -3 & 6 & -3 \\ 6 & -12 & 6 \\ -3 & 6 & -3 \end{bmatrix}$$

Any square matrix A has an adjugate matrix $A^{+} = \mathrm{adj}(A)$ and a determinant $|A| = \det(A)$. Such a matrix also has an inverse $A^{-1} = \mathrm{inv}(A)$ that can be computed as:

$$A^{-1} = \frac{A^{+}}{|A|}$$

or:

$$\mathrm{inv}(A) = \frac{\mathrm{adj}(A)}{\det(A)}$$

except in the case where $|A| = \emptyset$. In such a case, the matrix A is called <u>singular</u>.

<u>Example</u>:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$|A| = 1 \cdot \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \cdot \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \cdot \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

$$= 1 \cdot (-3) - 2 \cdot (-6) + 3 \cdot (-3)$$

$$= -3 + 12 - 9 \equiv \emptyset$$

$\Rightarrow$ A is singular. It does not possess in inverse.

Linear systems with singular matrices usually don't have a solution:

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

$$\Rightarrow |A| = 4 - 4 = 0$$

Let us look at the problem:

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$\Rightarrow \left| \begin{array}{l} x_1 + 2x_2 = 3 \\ 2x_1 + 4x_2 = 5 \end{array} \right|$$

Multiply the 1st equation with 2:

$$2x_1 + 4x_2 = 6$$

is in contradiction with the second equation.

If a linear system with a singular matrix does have a solution, the solution is not completely determined.

Let us look at the problem:

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

$$\Rightarrow \left| \begin{array}{l} x_1 + 2x_2 = 3 \\ 2x_1 + 4x_2 = 6 \end{array} \right|$$

The second equation is redundant.
Any solution with:

$$x_1 = 3 - 2x_2$$

is a good solution, e.g.

$$x_1 = x_2 = 1$$

<u>or</u>: $x_1 = -1$; $x_2 = 2$

<u>etc</u>.