3) Linear ODE with unknown inputs

$$\dot{\underline{x}} = A \cdot \underline{x} + B \cdot \underline{u} \quad ; \begin{cases} \underline{x}(t=\emptyset_+) = \underline{x}_0 \\ \underline{u}(t) = \emptyset \; ; \; \forall t < \emptyset \end{cases}$$

$$\underline{x}(t) = \underline{x}_p(t) + \underline{x}_h(t)$$

Homogeneous System:

$$\dot{\underline{x}}_h = A \cdot \underline{x}_h \quad ; \quad A \in \mathbb{R}^{n \times n}$$

Ansatz:

$$\underline{x}_h(t) = e^{At} \cdot \underline{c}_0$$

where:

$$e^{At} := I^{(n)} + At + \frac{A^2 t^2}{2!} + \frac{A^3 t^3}{3!} + \dots$$

$:=$ is defined as

Verification:

$$\frac{d}{dt}\left\{e^{At}\right\} = \phi^{(n)} + A + A^2t + \frac{A^3t^2}{2!} + \frac{A^4t^3}{3!} + \dots$$

$$= A\cdot\left[I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \dots\right]$$

$$= A\cdot e^{At}$$

$$= \left[I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \dots\right]\cdot A$$

$$= e^{At}\cdot A$$

$$\Rightarrow \frac{d}{dt}\left\{e^{At}\right\} \equiv A\cdot e^{At} \equiv e^{At}\cdot A$$

Notice: $A\cdot e^{At} \equiv e^{At}\cdot A$

although usually: $A\cdot B \neq B\cdot A$ for matrices.

Warning:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$e^{At} \neq \begin{bmatrix} e^{t} & e^{2t} \\ e^{3t} & e^{4t} \end{bmatrix}$$

The exponential of a matrix is <u>not</u> equal the matrix of exponentials.

$$\underline{\dot{x}}_h = A \cdot \underline{x}_h$$

$$\underline{x}_h = e^{At} \cdot \underline{c}_0$$

$$\Rightarrow \underline{\dot{x}}_h = A \cdot e^{At} \cdot \underline{c}_0$$

$$\Rightarrow A \cdot e^{At} \cdot \underline{c}_0 \overset{?}{=} A \cdot (e^{At} \cdot \underline{c}_0)$$

$\underline{\checkmark}$ q.e.d.

Particular Solution of Inhomogeneous System:

$$\underline{\dot{x}}_p = A \cdot \underline{x}_p + B \cdot \underline{u}$$

Ansatz:

$$\underline{x}_p(t) = e^{At} \cdot \underline{\gamma}(t)$$

$$\Rightarrow \underline{\dot{x}}_p(t) = A \cdot e^{At} \cdot \underline{\gamma}(t) + e^{At} \cdot \underline{\dot{\gamma}}(t)$$

$$\Rightarrow \cancel{A \cdot e^{At} \cdot \underline{\gamma}(t)} + e^{At} \cdot \underline{\dot{\gamma}}(t) \overset{?}{=} \cancel{A \cdot e^{At} \cdot \underline{\gamma}(t)} + B \cdot \underline{u}(t)$$

$$\Rightarrow e^{At} \cdot \underline{\dot{\gamma}}(t) = B \cdot \underline{u}(t)$$

$$\Rightarrow \dot{\underline{\gamma}}(t) = e^{-At} \cdot B \cdot \underline{u}(t)$$

(*without proof*: $\left\{e^{At}\right\}^{-1} \equiv e^{-At}$)

$$\Rightarrow \underline{\gamma}(t) = \int_{0-}^{t} e^{-A\tau} \cdot B \cdot \underline{u}(\tau)\, d\tau$$

$$\Rightarrow \underline{x}_p(t) = e^{At} \cdot \int_{0-}^{t} e^{-A\tau} \cdot B \cdot \underline{u}(\tau)\, d\tau$$

$$= \int_{0^-}^{t} e^{A(t-\tau)} \cdot B \cdot \underline{u}(\tau)\, d\tau$$

(*without proof*: $e^{At} \cdot e^{-A\tau} \equiv e^{A(t-\tau)}$)

$$\Rightarrow \underline{x}(t) = e^{At} \cdot \underline{c}_0 + \int_{0-}^{t} e^{A(t-\tau)} \cdot B \cdot \underline{u}(\tau)\, d\tau$$

Initial Conditions:

$$\underline{x}(t = 0_+) = \underline{x}_0 = \underline{c}_0 + \underbrace{\int_{0-}^{0+} e^{A(t-\tau)} \cdot B \cdot \underline{u}(\tau)\, d\tau}_{= 0}$$

unless $\underline{u}(t)$ contains Dirac at $t = 0$

$$\Rightarrow \underline{x}_0 = \underline{c}_0$$

$$\Rightarrow \underline{x}(t) = \underbrace{e^{At} \cdot \underline{x}_0}_{\substack{\text{state} \\ \text{response}}} + \underbrace{\int_{0-}^{t} e^{A(t-\tau)} \cdot B \cdot \underline{u}(\tau)\, d\tau}_{\substack{\text{input} \\ \text{response}}}$$