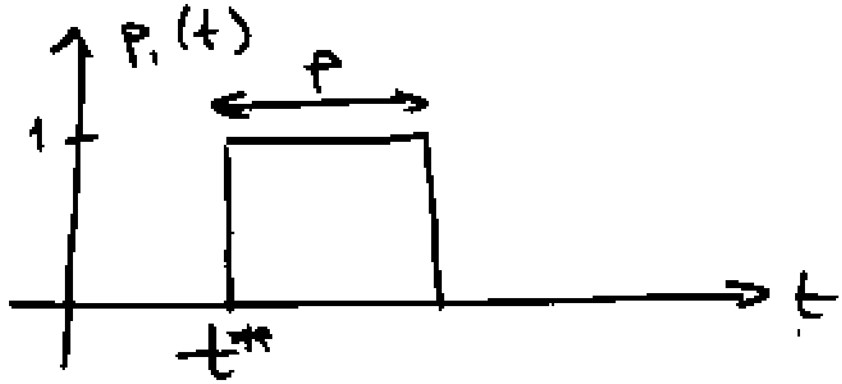
Analysis of Sampled Data:

A pulse can be mathematically described as:



$$p_1(t) = \varepsilon(t-t^*) - \varepsilon(t-t^*-p)$$

A pulse train starting at time zero can be written as an infinite sum of such pulses:

$$p_2(t) = \sum_{k=0}^{\infty} \left\{ \varepsilon(t-kT) - \varepsilon(t-kT-p) \right\} \; ; \; p<T$$

Sometimes, it is preferable to let the pulse train start at time $-\infty$:

$$p_3(t) = \sum_{k=-\infty}^{+\infty} \left\{ \varepsilon(t-kT) - \varepsilon(t-kT-p) \right\}; \; p<$$

$p_3(t)$ is periodic in T, hence we can develop into a Fourier series:

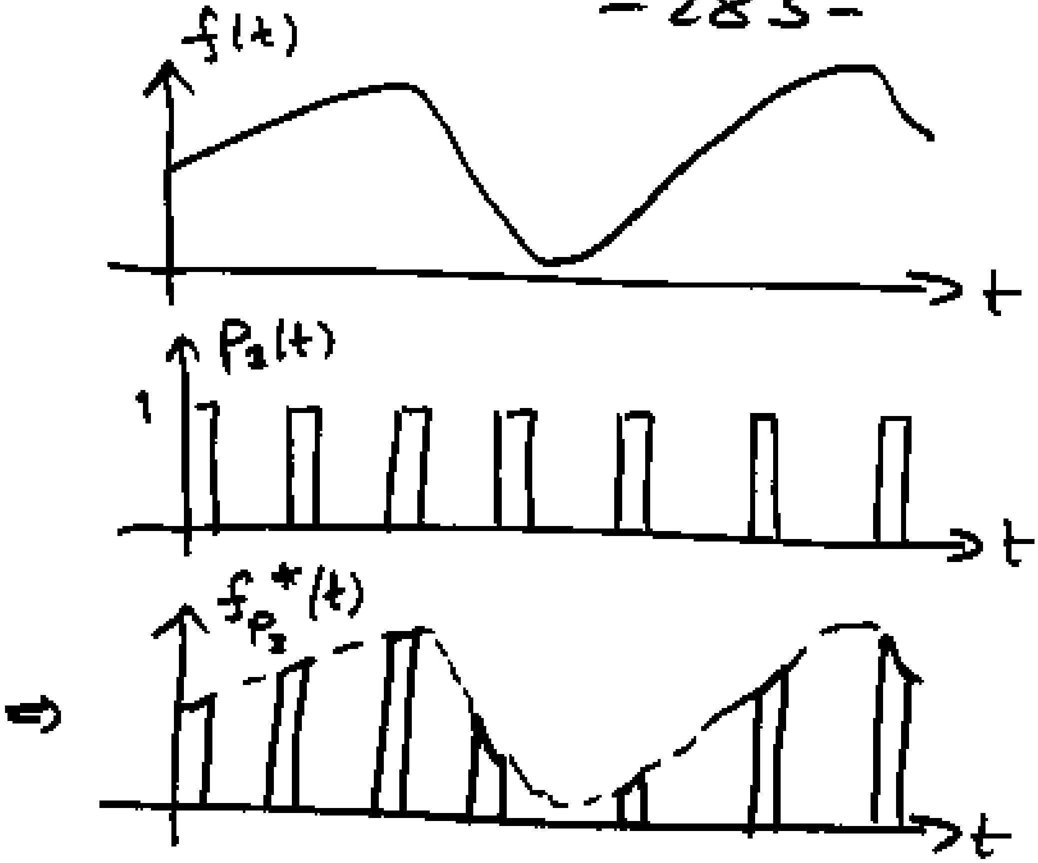
$$p_3(t) = \sum_{n=-\infty}^{+\infty} C_n e^{jn\omega_s t} \quad ; \; \omega_s = \frac{2\pi}{T}$$

$$C_n = \frac{1}{T} \int_0^T p_3(\tau) \cdot e^{-jn\omega_s \tau} d\tau$$

$$= \frac{1}{T} \int_0^p e^{-jn\omega_s \tau} d\tau = \frac{1 - e^{-jn\omega_s p}}{jn\omega_s T}$$

$$= \left(\frac{p}{T}\right) \cdot \operatorname{sinc}\left(n\omega_s \frac{p}{2}\right) \cdot e^{-jn\omega_s \frac{p}{2}}$$

$$\Rightarrow p_3(t) = \frac{p}{T} \cdot \sum_{n=-\infty}^{+\infty} \operatorname{sinc}\left(n\omega_s \frac{p}{2}\right) \cdot e^{jn\omega_s (t - \frac{p}{2})}$$



$$f_{p_3}^*(t) = f(t)\cdot p_3(t) = \sum_{n=-\infty}^{+\infty} C_n f(t)\cdot e^{jn\omega_s t}$$

$f_{p_3}^*(t)$ is non-periodic. It has a Fourier transform:

$$F_{p_3}^*(j\omega) = \int_{-\infty}^{+\infty} f_{p_3}^*(\tau)\, e^{-j\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{+\infty} \left(\sum_{n=-\infty}^{+\infty} C_n \cdot f(\tau)\cdot e^{jn\omega_s\tau} \right) \cdot e^{-j\omega\tau}\, d\tau$$

$$= \sum_{n=-\infty}^{+\infty} C_n \left\{ \int_{-\infty}^{+\infty} f(\tau) e^{jn\omega_s\tau} \cdot e^{-j\omega\tau}\, d\tau \right\}$$

Assuming that $f(t)$ has a Fourier transform, it will be:

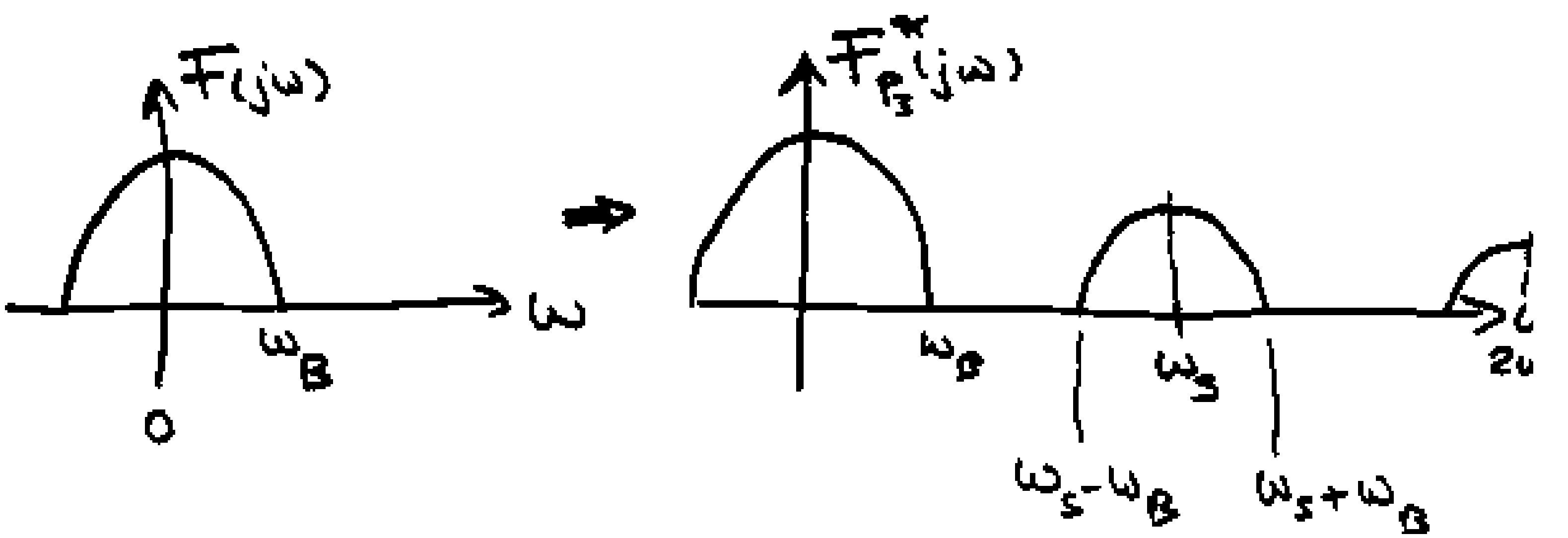
$$F(j\omega) = \mathcal{F}\{f(t)\} = \int_{-\infty}^{+\infty} f(\tau)\cdot e^{-j\omega\tau}\,d\tau$$

By the shifting theorem:

$$\int_{-\infty}^{+\infty} f(\tau) e^{-j\omega\tau}\cdot e^{jn\omega_s\tau}\,d\tau = F(j\omega - jn\omega_s)$$

$$\Rightarrow F_{p_1}^{*}(j\omega) = \sum_{n=-\infty}^{+\infty} C_n \cdot F(j\omega - jn\omega_s)$$

Assuming that $F(j\omega)$ is bandlimited, we can draw $F_p^{*}(j\omega)$ as follows:



⇒ A lowpass filter can retrieve the original signal iff

$$\omega_s \geq 2\omega_B$$

Shannon's Sampling Theorem.

This provides an upper limit on the sampling rate:

$$T = \frac{2\pi}{\omega_s} \leq \frac{\pi}{\omega_B}$$

⇒ The sampling rate is now bounded from below and from above:

$$\underbrace{T_{min}}_{\substack{\text{Dictated by} \\ \text{technology} \\ \text{in use}}} \leq T \leq \underbrace{T_{max}}_{\substack{\text{Dictated by} \\ \text{Systems theory} \\ \text{(information/stability)}}}$$

In practice, it may be more convenient to work with $p_2(t)$ and Laplace transform:

$$F_{p_2}^*(s) = \mathcal{L}\left\{f_{p_2}^*(t)\right\} = \mathcal{L}\left\{f(t)\cdot p_2(t)\right\}$$

$$\Rightarrow F_{p_2}^*(s) = F(s) * P_2(s)$$

↑ convolution

$$P_2(t) = \sum_{k=0}^{\infty}\left\{\varepsilon(t-kT) - \varepsilon(t-kT-p)\right\}$$

$$\Rightarrow P_2(s) = \sum_{k=0}^{\infty}\left\{\frac{1}{s}e^{-kTs} - \frac{1}{s}e^{-(kT+p)s}\right\}$$

$$= \sum_{k=0}^{\infty}\frac{1-e^{-ps}}{s}\cdot e^{-kTs}$$

$$= \frac{1-e^{-ps}}{s}\underbrace{\left[1+e^{-Ts}+e^{-2Ts}+\cdots\right]}_{H(s)}$$

$$H(s) = 1 + e^{-Ts} + e^{-2Ts} + e^{-3Ts} + \cdots$$

$$\Rightarrow e^{-Ts} \cdot H(s) = e^{-Ts} + e^{-2Ts} + e^{-3Ts} + \cdots$$

$$\Rightarrow \left(1 - e^{-Ts}\right) \cdot H(s) = 1$$

$$\Rightarrow H(s) = \frac{1}{1 - e^{-Ts}}$$

$$\Rightarrow P_2(s) = \frac{1 - e^{-ps}}{s\left(1 - e^{-Ts}\right)}$$

$$f_{p_2}^{*}(t) = \sum_{k=0}^{\infty} f(t) \left\{ \epsilon(t - kT) - \epsilon(t - kT - p) \right\}$$

$$\cong \sum_{k=0}^{\infty} f(kT) \cdot \left\{ \epsilon(t - kT) - \epsilon(t - kT - p) \right\}$$

; $p \ll T$.

$$\Rightarrow F_{p_2}^{*}(s) \cong \sum_{k=0}^{\infty} \frac{1 - e^{-ps}}{s} \cdot f(kT) \cdot e^{-kTs}$$

$$= \frac{1 - e^{-ps}}{s} \cdot \sum_{k=0}^{\infty} f(kT) \cdot e^{-kTs}$$

If an ideal sampler would have been used:

$$f^*(t) = \sum_{k=0}^{\infty} f(kT) \cdot \delta(t-kT)$$

$$\Rightarrow F^*(s) = \sum_{k=0}^{\infty} f(kT) \cdot e^{-Ts}$$

$$\Rightarrow F^*_{p_3}(s) \cong \frac{1-e^{-ps}}{s} \cdot F^*(s)$$

If $p \ll T$:

$$1-e^{-ps} = 1 - \underbrace{\left\{1 - ps + \frac{p^2s^2}{2} - \frac{p^3s^3}{6} + - \right.}_{\text{Taylor series}}$$

$$\cong 1 - 1 + ps = ps$$

$$\Rightarrow F^*_{p_3}(s) \cong p \cdot F^*(s)$$

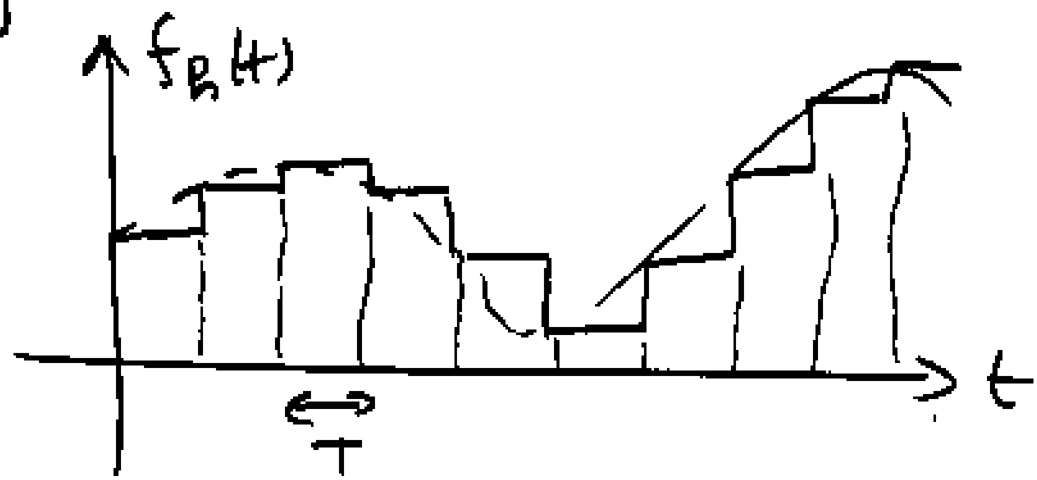
Notice that in the limit $p \to 0$:

$$\lim_{p \to 0} F_{p_z}^*(s) = \phi \neq F^*(s)$$

This is because the amplitude of the pulses was left at 1, rather than $\frac{1}{p}$, as the ideal sampler would.

($\delta(t)$ are pulses of infinitely small width, but infinitely tall amplitude.)

Let us now look at the signal $f_h(t)$:



$$f_h(t) = \sum_{k=0}^{\infty} f(kT)\left\{ \varepsilon(t-kT) - \varepsilon(t-(k+1)T) \right.$$

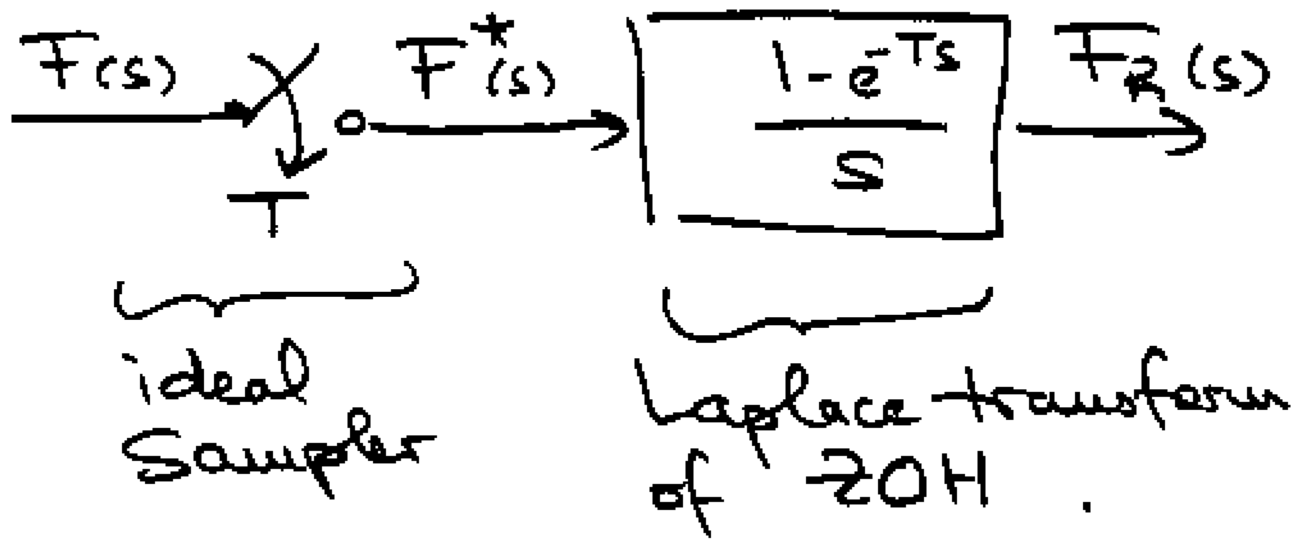
$$\Rightarrow F_h(s) = \sum_{k=0}^{\infty} f(kT)\cdot\left\{ \frac{1}{s} e^{-kTs} - \frac{1}{s} e^{-(k+1)Ts} \right.$$

$$= \sum_{k=0}^{\infty} f(kT)\cdot\frac{1}{s}\cdot\left(1-e^{-Ts}\right)\cdot e^{-kTs}$$

$$= \frac{1-e^{-Ts}}{s}\cdot\underbrace{\sum_{k=0}^{\infty} f(kT)\cdot e^{-kTs}}_{F^*(s)}$$

$$\Rightarrow F_h(s) = \frac{1-e^{-Ts}}{s} \cdot F^*(s)$$

Graphically:



Relationship Between $F(s)$ and $F^*(s)$

Example: $F(s) = \frac{1}{s} \rightarrow f(t) = \varepsilon(t)$

$$\Rightarrow f^*(t) = \sum_{k=0}^{\infty} f(kT) \cdot \delta(t-kT)$$

$$= \sum_{k=0}^{\infty} \delta(t-kT)$$

$$\Rightarrow F^*(s) = \sum_{k=0}^{\infty} e^{-kTs} = \frac{1}{1-e^{-Ts}}$$

Example: $F(s) = \frac{1}{s+a} \Rightarrow f(t) = e^{-at} \cdot \varepsilon(t)$

$$\Rightarrow f^*(t) = \sum_{k=0}^{\infty} e^{-akT} \cdot \delta(t-kT)$$

$$\Rightarrow F^*(s) = \sum_{k=0}^{\infty} e^{-kT(s+a)} = \frac{1}{1-e^{-T(s+a)}}$$

<u>Example</u>: $F(s) = \dfrac{1}{s(s+1)}$

We use PFE:

$$F(s) = \frac{1}{s} - \frac{1}{s+1}$$

$$\Rightarrow f(t) = \varepsilon(t) - e^{-t} \cdot \varepsilon(t)$$

$$\Rightarrow F^*(s) = \frac{1}{1-e^{-Ts}} - \frac{1}{1-e^{-T(s+1)}}$$

$$= \frac{\left(1-e^{-T} \cdot e^{-Ts}\right) - \left(1-e^{-Ts}\right)}{\left(1-e^{-Ts}\right) \cdot \left(1-e^{-T} \cdot e^{-Ts}\right)}$$

$$= \frac{\left(1-e^{-T}\right) \cdot e^{-Ts}}{\left(1-e^{-Ts}\right) \cdot \left(1-e^{-T} \cdot e^{-Ts}\right)}$$

Let us introduce a new variable:

$$z = e^{Ts}$$

$$\Rightarrow e^{-Ts} = z^{-1}$$

$$\Rightarrow F(z) = \frac{(1-e^{-T})\,z^{-1}}{(1-z^{-1})\cdot(1-e^{-T}\cdot z^{-1})}$$

$$= \frac{(1-e^{-T})\cdot z}{(z-1)(z-e^{-T})}$$

$$= \frac{(1-e^{-T})\cdot z}{z^2-(1+e^{-T})\cdot z+e^{-T}}$$

Rational function in z.
This is the Z-Transform.

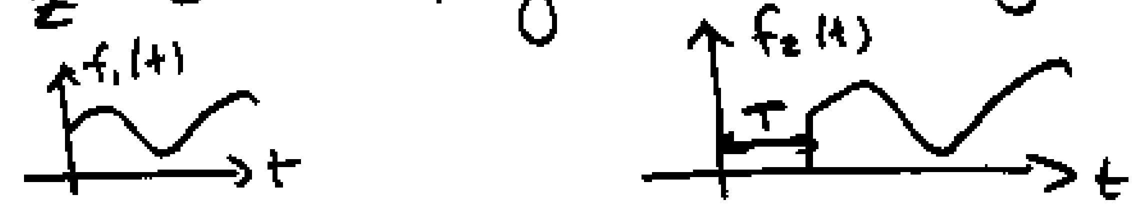
Physical interpretation:

$s \hat{=}$ derivative ("$\frac{d}{dt}$")

$\frac{1}{s} = s^{-1} \hat{=}$ integral ("$\int$")

$z \hat{=}$ left shift by T

$\frac{1}{z} = z^{-1} \hat{=}$ right shift by T



$$f_1(t) \longrightarrow \boxed{z^{-1}} \longrightarrow f_2(t) = f_1(t-T)$$