Spectral Decomposition:

In the last section, we solved the eigenvalue/eigenvector equation:

$$A \cdot V = V \cdot \Lambda$$

for Λ:

$$\boxed{\Lambda = V^{-1} \cdot A \cdot V}$$

However, we can also solve it for A:

$$\boxed{A = V \cdot \Lambda \cdot V^{-1}}$$

This is called the spectral decomposition of the square matrix A. It is very useful for many purposes.

Polynomials in A:

Let us compute A^2:

$$A^2 = (V \cdot \Lambda \cdot V^{-1})^2$$
$$= (V \cdot \Lambda \cdot V^{-1}) \cdot (V \cdot \Lambda \cdot V^{-1})$$
$$= V \cdot \Lambda \cdot (V^{-1} \cdot V) \cdot \Lambda \cdot V^{-1}$$
$$= V \cdot \Lambda \cdot I^{(n)} \cdot \Lambda \cdot V^{-1}$$
$$= V \cdot \Lambda \cdot \Lambda \cdot V^{-1}$$
$$= V \cdot \Lambda^2 \cdot V^{-1}$$

Similarly:

$$A^n = V \cdot \Lambda^n \cdot V^{-1}$$

[This is how Matlab actually computes A^n.]

Transcendental Functions:

We have already seen the Taylor-series expansion of e^{At}:

$$e^{At} = I^{(n)} + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \dots$$

Until now, we computed e^{At} using:

$$\boxed{e^{At} = \mathcal{L}^{-1}\left\{ (sI^{(n)} - A)^{-1} \right\}}$$

The spectral decomposition offers a new way to compute e^{At}:

$$e^{At} = V \cdot I^{(n)} \cdot V^{-1} + V \cdot \Lambda \cdot t \cdot V^{-1} + \frac{V \cdot \Lambda^2 t^2 \cdot V^{-1}}{2!} + \dots$$

$$= V \left[I^{(n)} + \Lambda t + \frac{\Lambda^2 t^2}{2!} + \frac{\Lambda^3 t^3}{3!} + \dots \right] V^{-1}$$

$$\Rightarrow \boxed{e^{At} = V \cdot e^{\Lambda t} \cdot V^{-1}}$$

Since $\Lambda = \begin{bmatrix} \lambda_1 & & & \emptyset \\ & \lambda_2 & & \\ & & \ddots & \\ \emptyset & & & \lambda_n \end{bmatrix}$

is a diagonal matrix (if all eigenvalues are distinct),

$$\Rightarrow e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & & \emptyset \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ \emptyset & & & e^{\lambda_n t} \end{bmatrix}$$

can be written at once.

[This is how Matlab computes expm(A*t).]

The concept can be generalized to all functions that have Taylor-series expansions.

For example:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + - \cdots$$

$$\Rightarrow \sin(A) = A - \frac{A^3}{3!} + \frac{A^5}{5!} - \frac{A^7}{7!} + - \cdots$$

can be computed as:

$$\boxed{\sin(A) = V \cdot \sin(\Lambda) \cdot V^{-1}}$$

Obviously:

$$\boxed{e^{jA} = \cos(A) + j \cdot \sin(A)}$$

works now also for matrices.

Functions that have a Taylor-series expansion are called transcendental functions.

Cayley-Hamilton Theorem:

Given a matrix A, e.g.

$$A = \begin{bmatrix} -1 & 1 & 2 \\ \phi & -2 & 1 \\ \phi & \phi & -3 \end{bmatrix}$$

Square matrices have a characteristic polynomial:

$$(\lambda+1)(\lambda+2)(\lambda+3)$$

$$= \lambda^3 + 6\lambda^2 + 11\lambda + 6$$

The roots of the characteristic polynomial are the eigenvalues of A. They can be found from the equation:

$$\lambda^3 + 6\lambda^2 + 11\lambda + 6 = \emptyset$$

The Cayley-Hamilton theorem states that every square matrix satisfies its own characteristic polynomial, i.e.:

$$A^3 + 6 \cdot A^2 + 11 \cdot A + 6 I^{(4)} = \emptyset$$

Proof:

$$V \left[\Lambda^3 + 6\Lambda^2 + 11\Lambda + 6 \cdot I^{(4)} \right] \cdot V^{-1} = \emptyset$$

$$\Rightarrow \Lambda^3 + 6V^2 + 11\Lambda + 6 \cdot I^{(4)} = \emptyset$$

However, since:

$$\Lambda = \begin{bmatrix} \lambda_1 & \emptyset & \emptyset \\ \emptyset & \lambda_2 & \emptyset \\ \emptyset & \emptyset & \lambda_3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} (\lambda_1^3+6\lambda_1^2+11\lambda_1+6) & \emptyset & \emptyset \\ \emptyset & (\lambda_2^3+6\lambda_2^2+11\lambda_2+6) & \emptyset \\ \emptyset & \emptyset & (\lambda_3^3+6\lambda_3^2+11\lambda_3+6) \end{bmatrix} = \emptyset$$

which is obviously true.

q.e.d.

$$\Rightarrow \boxed{A^3 = -6 \cdot I^{(n)} - 11 \cdot A - 6 \cdot A^2}$$

A square matrix $A \in \mathbb{R}^{n \times n}$ has the property that A^n is linearly dependent on powers of $A < n$.

We can expand the previous equation by A:

$$A^4 = -6A - 11 \cdot A^2 - 6 \cdot A^3$$

<u>etc</u>.

$\Rightarrow$ All higher powers of A can be computed as linear combinations of lower powers.