Eigenvalues and Eigenvectors:

Given a square matrix:

$$A \in \mathbb{R}^{n \times n}$$

Such a matrix has n eigenvalues λ_i and $\leq n$ (right) eigenvectors $\underline{v}_i$ that satisfy the equation:

$$A \cdot \underline{v}_i = \lambda_i \cdot \underline{v}_i$$

We want to exclude the trivial case:

$$\underline{v}_i = \emptyset$$

which satisfies any matrix.

We can expand:

$$A \cdot \underline{v}_i = \lambda_i \left(I^{(n)} \cdot \underline{v}_i \right)$$

$$= \left(\lambda_i \cdot I^{(n)} \right) \cdot \underline{v}_i$$

$$\Rightarrow \underbrace{(\lambda_i I^{(n)} - A)}_{M} \cdot \underbrace{\underline{v}_i}_{\underline{x}} = \underbrace{\underline{\phi}}_{\underline{b}}$$

We need to solve the linear equation system:

$$M \cdot \underline{x} = \underline{b}$$

Assuming that M is non-singular, we can expand:

$$M^{-1} \cdot (M \cdot \underline{x}) = M^{-1} \cdot \underline{b}$$

$$= (M^{-1} \cdot M) \cdot \underline{x} = I^{(n)} \cdot \underline{x} = \underline{x}$$

$$\Rightarrow \underline{x} = M^{-1} \cdot \underline{b} = M^{-1} \cdot \phi = \phi$$

$$\Rightarrow \boxed{\underline{v}_i = \phi}$$

This is the trivial solution that we excluded.

$\Rightarrow$ A non-trivial solution can only exist if M is singular, i.e. if $\det(M) = \emptyset$.

$$\boxed{\det\left(\lambda_i \cdot I^{(n)} - A\right) = \emptyset}$$

is the equation that provides the eigenvalues λ_i of A.

$\det(\lambda_i \cdot I - A)$ is a n^{th}-order polynomial in λ_i. The eigenvalues of A are the roots of that polynomial.

The polynomial $\det(\lambda_i I - A)$ is called the <u>characteristic</u> polynomial of the system.

Remember that:

$$G(s) = \underline{c}' \cdot (sI-A)^{-1} \cdot \underline{b} + d$$

$$= \underline{c}' \cdot \frac{(sI-A)^{+}}{|sI-A|} \cdot \underline{b} + d$$

$$= \frac{\underline{c}' \cdot (sI-A)^{+} \cdot \underline{b} + d \cdot |sI-A|}{|sI-A|}$$

$$= \frac{N(s)}{D(s)}$$

$\Rightarrow D(s) = \det(sI-A)$

The characteristic polynomial is the denominator polynomial of the transfer function.

$\Rightarrow$ The eigenvalues of the system matrix, A, are the poles of the transfer function $G(s)$.

Example:

$$A = \begin{bmatrix} 1248 & -788 & 112 \\ 2126 & -1342 & 190 \\ 986 & -620 & 83 \end{bmatrix}$$

$$\Rightarrow \quad sI - A = \begin{bmatrix} (s-1248) & 788 & -112 \\ -2126 & (s+1342) & -190 \\ -986 & 620 & (s-83) \end{bmatrix}$$

$$\Rightarrow \det(sI-A) = (s-1248)\cdot(s+1342)\cdot(s-83)$$
$$-(s-1248)\cdot(-190)\cdot 620 - 788\cdot(-2126)\cdot(s-83)$$
$$+ 788\cdot(-190)\cdot(-986) + (-112)\cdot(-2126)\cdot 620$$
$$-(-112)\cdot(s+1342)\cdot(-986)$$

$$= s^3 + 11s^2 + 38s + 40$$

$$= (s+2)(s+4)(s+5)$$

$$\Rightarrow \quad \lambda_1 = -2, \quad \lambda_2 = -4, \quad \lambda_3 = -5$$

In Matlab:

Given A as above:

$$\text{poly}(A) = \begin{bmatrix} 1 & 11 & 38 & 40 \end{bmatrix}$$

$$\text{roots}(\text{poly}(A)) = \begin{bmatrix} -2 \\ -4 \\ -5 \end{bmatrix}$$

$$\text{eig}(A) = \begin{bmatrix} -2 \\ -4 \\ -5 \end{bmatrix}$$

$$\Rightarrow \boxed{\text{eig}(A) \equiv \text{roots}(\text{poly}(A))}$$

In reality, Matlab uses a different method to compute eigenvalues. It finds roots of a polynomial by computing the eigenvalues of the <u>lower companion matrix</u>.

roots([1, 11, 38, 40])

⇒ Matlab internally computes the lower-companion matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -40 & -38 & -11 \end{bmatrix}$$

then computes its eigenvalues.

The lower-companion matrix is the system matrix of the controller-canonical form.

Eigenvectors:

$$(\lambda_i \cdot I^{(n)} - A) \cdot \underline{v}_i = \emptyset$$

The eigenvectors are only determined up to their length.

Proof:

$$\alpha\,(\lambda_i I - A) \cdot \underline{v}_i = \emptyset$$

for any $\alpha \neq \emptyset$
(of course also for $\alpha = \emptyset$).

$$\Rightarrow (\lambda_i I - A) \cdot \underbrace{(\alpha \cdot \underline{v}_i)}_{\hat{\underline{v}}_i} = \emptyset$$

$\hat{\underline{v}}_i$ is also an eigenvector.

$\Rightarrow$ We can normalize the length of the eigenvectors to 1.

Example:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

$$\Rightarrow \det(\lambda I - A) = \begin{vmatrix} \lambda & -1 & 0 \\ 0 & (\lambda+1) & -1 \\ 0 & 0 & (\lambda+2) \end{vmatrix}$$

$$= \lambda(\lambda+1)(\lambda+2)$$

$$\Rightarrow \begin{aligned} \lambda_1 &= 0 \\ \lambda_2 &= -1 \\ \lambda_3 &= -2 \end{aligned}$$

$$(\lambda_i I - A)\underline{v}_i = 0$$

$$\begin{bmatrix} \lambda_i & -1 & 0 \\ 0 & (\lambda_i+1) & -1 \\ 0 & 0 & (\lambda_i+2) \end{bmatrix} \cdot \begin{bmatrix} v_{1i} \\ v_{2i} \\ v_{3i} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left| \begin{aligned} &\lambda_i \cdot v_{1i} - v_{2i} = 0 \\ &(\lambda_i+1) \cdot v_{2i} - v_{3i} = 0 \\ &(\lambda_i+2) \cdot v_{3i} = 0 \end{aligned} \right|$$

$\underset{\sim}{\lambda_1 = \emptyset}$:

$$\left| \begin{array}{r} -V_{21} = \emptyset \\ V_{21} - V_{31} = \emptyset \\ 2V_{31} = \emptyset \end{array} \right|$$

$$\Rightarrow V_{21} \equiv V_{31} \equiv \emptyset$$

We can choose V_{11} freely, eg.
$V_{11} = 1$

$$\Rightarrow \underline{v}_1 = \begin{bmatrix} 1 \\ \emptyset \\ \emptyset \end{bmatrix} \; ; \; |\underline{v}_1| = 1$$

$\underset{\sim}{\lambda_2 = -1}$:

$$\left| \begin{array}{r} -V_{12} - V_{22} = \emptyset \\ -V_{32} = \emptyset \\ V_{32} = \emptyset \end{array} \right|$$

$$\Rightarrow V_{32} = \emptyset \; ; \; V_{22} = -V_{12}$$

We choose e.g. $V_{12} = 1 \Rightarrow V_{22} = -1$

$$\underline{v}_2 = \begin{bmatrix} 1 \\ -1 \\ \emptyset \end{bmatrix} \Rightarrow |\underline{v}_2| = \sqrt{1+1} = \sqrt{2}$$

Normalization:

$$\underline{v}_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ \emptyset \end{bmatrix}$$

$\lambda_3 = -2$:

$$\left| \begin{aligned} -2v_{13} - v_{23} &= \emptyset \\ -v_{23} - v_{33} &= \emptyset \\ \emptyset &= \emptyset \end{aligned} \right|$$

$\Rightarrow v_{23} = -2v_{13}$; $v_{33} = -v_{23}$

Let $v_{13} = 1$

$$\Rightarrow \underline{v}_3 = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \Rightarrow |\underline{v}_3| = \sqrt{9} = 3$$

$$\Rightarrow \underline{v}_3 = \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \end{bmatrix}$$

Without Proof:

(1) Single eigenvalues have always a single eigen-vector associated with them (except for normalization).

(2) Eigenvectors associated with different eigenvalues are always linearly independent of each other.

Multiple eigenvalues:

Example:

$$A = \begin{bmatrix} 1 & \phi & \phi \\ 1 & 1 & 1 \\ -1 & \phi & \phi \end{bmatrix}$$

$$\Rightarrow \det(\lambda I - A) = \left| \begin{bmatrix} (\lambda-1) & \phi & \phi \\ -1 & (\lambda-1) & -1 \\ 1 & \phi & \lambda \end{bmatrix} \right|$$

$$= \lambda(\lambda-1)^2 \quad \Rightarrow \quad \begin{array}{l} \lambda_1 = \phi \\ \lambda_2 = \lambda_3 = 1 \end{array}$$

$$(\lambda_i I - A)\cdot \underline{v}_i = \emptyset$$

$$\Rightarrow \begin{bmatrix} (\lambda_i - 1) & \emptyset & \emptyset \\ -1 & (\lambda_i - 1) & -1 \\ 1 & \emptyset & \lambda_i \end{bmatrix} \cdot \begin{bmatrix} v_{1i} \\ v_{2i} \\ v_{3i} \end{bmatrix} = \begin{bmatrix} \emptyset \\ \emptyset \\ \emptyset \end{bmatrix}$$

$$\Rightarrow \left| \begin{array}{l} (\lambda_i - 1)\cdot v_{1i} = \emptyset \\ -v_{1i} + (\lambda_i - 1)\cdot v_{2i} - v_{3i} = \emptyset \\ v_{1i} + \lambda_i \cdot v_{3i} = \emptyset \end{array} \right|$$

$\lambda_1 = \emptyset$:

$$\left| \begin{array}{l} -v_{11} = \emptyset \\ -v_{11} - v_{21} - v_{31} = \emptyset \\ v_{11} = \emptyset \end{array} \right|$$

$$\Rightarrow v_{11} = \emptyset \; ; \; v_{21} = -v_{31}$$

e.g. $\underline{v}_1 = \begin{bmatrix} \emptyset \\ 1 \\ -1 \end{bmatrix} \Rightarrow |\underline{v}_1| = \sqrt{2}$

$$\Rightarrow \underline{v}_1 = \begin{bmatrix} \emptyset \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$\underline{\lambda_2 = \lambda_3 = 1}$:
$$\left| \begin{array}{c} \emptyset = \emptyset \\ -V_{12} - V_{32} = \emptyset \\ V_{12} + V_{32} = \emptyset \end{array} \right|$$

$$\Rightarrow V_{32} = -V_{12}$$

e.g. $\underline{V}_2 = \begin{bmatrix} \emptyset \\ 1 \\ \emptyset \end{bmatrix}$; $\underline{V}_3 = \begin{bmatrix} 1 \\ \emptyset \\ -1 \end{bmatrix}$

Normalization :

$$\underline{V}_2 = \begin{bmatrix} \emptyset \\ 1 \\ \emptyset \end{bmatrix} \quad ; \quad \underline{V}_3 = \begin{bmatrix} 1/\sqrt{2} \\ \emptyset \\ -1/\sqrt{2} \end{bmatrix}$$

$\underline{V}_2$ and $\underline{V}_3$ are linearly independent of each other. Both are also linearly independent of $\underline{V}_1$.

In this example, the double eigenvalue led to two separate linearly independent eigenvectors.

This is not always the case.

Example:

$$A = \begin{bmatrix} 1 & 1 \\ \emptyset & 1 \end{bmatrix} \Rightarrow \lambda I - A = \begin{bmatrix} (\lambda-1) & -1 \\ \emptyset & (\lambda-1) \end{bmatrix}$$

$$\Rightarrow \det(\lambda I - A) = (\lambda-1)^2$$

$$\Rightarrow \underline{\underline{\lambda_1 = \lambda_2 = 1}}$$

$$\begin{bmatrix} (\lambda_i-1) & -1 \\ \emptyset & (\lambda_i-1) \end{bmatrix} \cdot \begin{bmatrix} v_{1i} \\ v_{2i} \end{bmatrix} = \begin{bmatrix} \emptyset \\ \emptyset \end{bmatrix}$$

$$\Rightarrow \left| \begin{array}{l} (\lambda_i-1)\cdot v_{1i} - v_{2i} = \emptyset \\ (\lambda_i-1)\cdot v_{2i} = \emptyset \end{array} \right|$$

$\underset{\sim}{\lambda_1 = \lambda_2 = 1}$: $\left| \begin{array}{l} -v_{21} = \phi \\ \phi = \phi \end{array} \right|$

$$\Rightarrow v_{21} = \phi$$

$$\Rightarrow \underline{v}_1 = \begin{bmatrix} 1 \\ \phi \end{bmatrix}$$

is an eigenvector. There is no second eigenvector that is linearly independent of $\underline{v}_1$.

Let ν_i be the number of eigenvectors associated with the eigenvalue λ_i of multiplicity m_i:

$$\boxed{\begin{array}{l} m_i = \text{multiplicity}\{\lambda_i\} \\ \Rightarrow \nu_i \leq m_i \end{array}}$$