# Transformation to Controller-canonical form:

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**Algorithm (without proof):**

(1) We compute the so-called controllability matrix:

$$Q_c = [\underline{b}, A\underline{b}, A^2\underline{b}, \ldots, A^{n-1}\underline{b}]$$

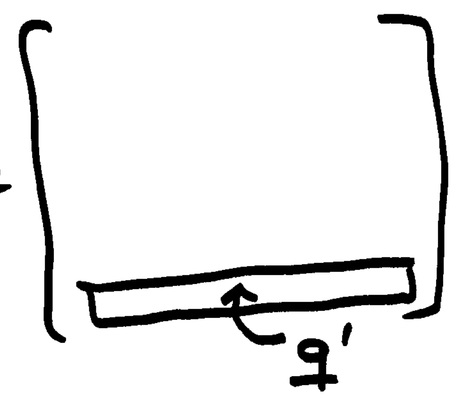
$$Q_c \in \mathbb{R}^{n \times n}$$

(2) We compute its inverse:

$$Q_c^{-1} = \mathrm{inv}(Q_c).$$

(3) We extract the last row of $Q_c^{-1}$, called $q'$:

$Q_c^{-1} =$



(4) We build the matrix:

$$T = [\underline{q}'; \underline{q}' \cdot A; \underline{q}' \cdot A^2; \dots; \underline{q}' \cdot A^{n-1}]$$

← Matlab notation

(5) We use $T$ in a similarity transformation:

$$\hat{A} = T \cdot A / T$$
$$\underline{\hat{b}} = T \cdot \underline{b}$$
$$\underline{\hat{c}}' = \underline{c}' / T$$
$$\hat{d} = d$$

← Matlab notation

$\Rightarrow$ The resulting representation will be in controller-canonical form.

Example:

$$\left|\begin{aligned} \dot{\underline{x}} &= \begin{bmatrix} 73 & -31 \\ 184 & -78 \end{bmatrix} \underline{x} + \begin{bmatrix} 2 \\ 5 \end{bmatrix} u \\ y &= \begin{bmatrix} -9 & 4 \end{bmatrix} \underline{x} \end{aligned}\right|$$

$$A \cdot \underline{b} = \begin{bmatrix} -9 \\ -22 \end{bmatrix} \Rightarrow Q_c = \left[\begin{array}{c:c} 2 & -9 \\ 5 & -22 \end{array}\right]$$

$$\det(Q_c) = 1$$

$$\Rightarrow Q_c^{-1} = Q_c^{+} = \begin{bmatrix} -22 & 9 \\ -5 & +2 \end{bmatrix}$$

$$\Rightarrow \underline{g}' = \begin{bmatrix} -5 & 2 \end{bmatrix}$$

$$\Rightarrow \underline{g}' \cdot A = \begin{bmatrix} 3 & -1 \end{bmatrix}$$

$$T = \left[\begin{array}{cc} -5 & 2 \\ \hdashline 3 & -1 \end{array}\right] \Rightarrow T^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$$

$$\Rightarrow \hat{A} = T \cdot A \cdot T^{-1} = \begin{bmatrix} 0 & 1 \\ -10 & -5 \end{bmatrix}$$

$$\underline{\hat{b}} = T \cdot \underline{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\underline{\hat{c}}' = \underline{c}' \cdot T^{-1} = [3 \quad 2]$$

$$\Rightarrow \left| \begin{array}{l} \dot{\underline{\underline{\xi}}} = \begin{bmatrix} 0 & 1 \\ -10 & -5 \end{bmatrix} \underline{\underline{\xi}} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y = [3 \quad 2] \, \underline{\underline{\xi}} \end{array} \right|$$

$$\Rightarrow \; G(s) = \frac{2s + 3}{s^2 + 5s + 10}$$

The algorithm can be conveniently programmed in Matlab:

```
A=[73, -31; 184, -78];
b=[2;5];
c=[-9, 4];
Qc=[b, A*b]
Qcin = inv(Qc)
q = Qcin(2,:)
T= [q; q*A]
Ah= T*A/T
bh = T*b
ch= c/T
```

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Matlab offers a (numerically better!) built-in algorithm to accomplish the same:

```
A = [73, -31; 184, -78];
b = [2; 5];
c = [-9, 4];
d = 0;
S = ss(A, b, c, d)
G = tf(S)
[N, D] = tfdata(G, 'v')
```

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Warnings:

(1) $Q_c^{-1}$ may not exist. In this situation, the algorithm fails.

⇒ cf. ECE 441

(2) This algorithm is numerically ill-conditioned. It works well only for small systems (up to $n=6$).

⇒ cf. ECE 544

# Transformation to Observer-canonical form:

## Algorithm (without proof):

(1) You build the so-called observability matrix:

$$Q_o = [\underline{c}' ; \underline{c}'A ; \underline{c}'A^2 ; \ldots ; \underline{c}'A^{n-1}]$$

$$Q_o \in \mathbb{R}^{n \times n}$$

(2) We compute its inverse:

$$Q_o^{-1} = \text{inv}(Q_o)$$

(3) We extract the last column.

$$\underline{q} = Q_o^{-1}(:, n)$$

$$Q_0^{-1} = \begin{bmatrix} \quad & \quad & \Big| \; \end{bmatrix} \; \underline{q}$$

(4) We build the matrix:

$$P = [\, \underline{q} \,,\, A \cdot \underline{q} \,,\, A^2 \cdot \underline{q} \,, \ldots ,\, A^{n-1} \cdot \underline{q} \,]$$

(5) We take its inverse:

$$T = \text{inv}(P)$$

(6) We use $T$ in a similarity transformation.

$\Rightarrow$ The resulting representation will be in observer-canonical form.

<u>Warnings</u>:
- $Q_0^{-1}$ may not exist.
- This algorithm is just as badly conditioned as the previous one.

$\Rightarrow$ It may make sense to compute <u>both</u> algorithms so that one gets a feel for the accumulated numerical garbage.