Routh-Hurwitz Stability Criterion:

(1) $D(s)$ has all poles in LHP iff all elements in the first column of the Routh scheme have the same sign.

(2) The number of poles in RHP equals the number of sign changes in the first column of the Routh scheme.

Example:

$$D(s) = s^4 + 2s^3 + 3s^2 + 4s + 5$$

	s^4	1	3	5
	s^3	2	4	
sign change	s^2	1	5	
sign change	s^1	-6		
	s^0	5		

$\Longrightarrow$ 2 poles in RHP.

Example:

$$D(s) = s^5 + 2s^4 + 4s^3 + 8s^2 + 3s + 2$$

s^5	1	4	3
s^4	2	8	2
s^3	$\emptyset$	2	
s^2	(?)		
s^1			
s^0			

The algorithm fails if a $\emptyset$ shows up in the first column.

Modified Algorithm (Enhancement).

If a $\emptyset$ shows up in the first column, replace $\emptyset$ by a small number ϵ and continue. Afterwards replace ϵ by $\emptyset^+$ and by $\emptyset^-$, and determine the number of sign changes in both cases.

Two things can happen:

(a) The number of sign changes is the same in both cases

⇒ This is the number of poles in the right half plane.

(b) The number of sign changes is different.

⇒ The smaller of the two determines the # of poles in the right half plane, and the difference between the two determines the # of poles on the imaginary axis.

Example continued:

$$D(s) = s^5 + 2s^4 + 4s^3 + 8s^2 + 3s + 2$$

$$
\begin{array}{c|ccc}
s^5 & 1 & 4 & 3 \\
s^4 & 2 & 8 & 2 \\
s^3 & \varepsilon & 2 & \\
s^2 & \dfrac{8\varepsilon-4}{\varepsilon} & 2 & \\
s^1 & \dfrac{-2\varepsilon^2-16\varepsilon-8}{8\varepsilon-4} & & \\
s^0 & 2 & &
\end{array}
$$

When $\varepsilon \longrightarrow \emptyset$:

$$
\begin{array}{c|ccc}
s^5 & 1 & 4 & 3 \\
s^4 & 2 & 8 & 2 \\
s^3 & \varepsilon & 2 & \\
s^2 & -4/\varepsilon & 2 & \\
s^1 & 2 & & \\
s^0 & 2 & &
\end{array}
$$

$\varepsilon \to \emptyset^+$: $(+\,+\,+\,-\,+\,+)$ $\Rightarrow$ 2 sign changes

$\varepsilon \to \emptyset^-$: $(+\,+\,-\,+\,+\,+)$ $\Rightarrow$ 2 sign changes

$\Rightarrow$ $\underline{\underline{\text{2 poles in RHP}}}$

Example:

$$D(s) = s^3 + 2s^2 + s + 2$$

s^3	1	1
s^2	2	2
s^1	ε	
s^0	2	

$\varepsilon \to \emptyset^+$: $(+ + + +)$ $\Rightarrow$ no sign changes
$\varepsilon \to \emptyset^-$: $(+ + - +)$ $\Rightarrow$ 2 sign changes

$\Longrightarrow$ 2 poles on imaginary axis.

Alternate Approach:

Given:

$$D(s) = 5(s+2)(s-1+3j)(s-1-3j)$$

Has its poles at

$$s_1 = -2$$
$$s_2 = +1 - 3j$$
$$s_3 = +1 + 3j$$

Let us look at a <u>different</u> polynomial, where s got replaced by 1/s :

$$\hat{D}\left(\tfrac{1}{s}\right) = 5\left(\tfrac{1}{s}+2\right)\left(\tfrac{1}{s}-1+3j\right)\left(\tfrac{1}{s}-1-3j\right)$$

$$= \frac{5}{s^3}\left(1+2s\right)\cdot\left(1+(-1+3j)s\right)\left(1+(-1-3j)s\right)$$

$$= \frac{5\cdot 2\cdot(-1+3j)\cdot(-1-3j)}{s^3}\left(\frac{1}{2}+s\right)\left(\frac{1}{-1+3j}+s\right)\left(\frac{1}{-1-3j}+s\right)$$

$$= \frac{100}{s^3}\left(s+\frac{1}{2}\right)\left(s+\frac{-1-3j}{10}\right)\left(s+\frac{-1+3j}{10}\right)$$

$$= \frac{100}{s^3}\left(s+\frac{1}{2}\right)\left(s-0.1-0.3j\right)\left(s-0.1+0.3j\right)$$

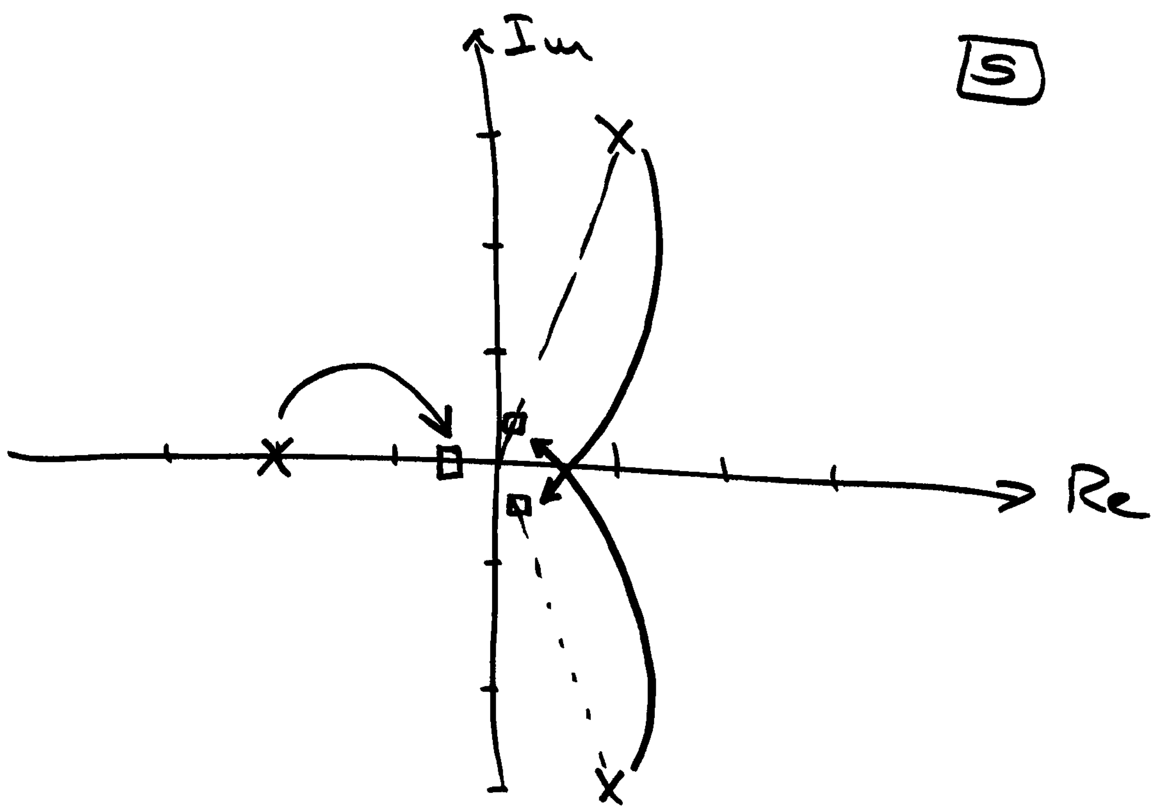
$\hat{D}(s)$ has its zeros at:

$$\hat{s}_1 = -0.5$$
$$\hat{s}_2 = +0.1 + 0.3j$$
$$\hat{s}_3 = +0.1 - 0.3j$$

There is a direct relation between the roots of $D(s)$ and those of $\hat{D}(s)$:

$$|\hat{s}_i| = \frac{1}{|s_i|}$$

$$\angle \hat{s}_i = -\angle s_i$$



Certainly, the numbers of poles in the RHP of $D(s)$ and $\hat{D}(s)$ are the same.

Hence we can exchange $s \to \frac{1}{s}$ before applying the Routh scheme. The answers must remain the same.

Given:

$$D(s) = a_n \cdot s^n + a_{n-1} \cdot s^{n-1} + \cdots + a_1 s + a_0$$

$$\Rightarrow \hat{D}\left(\tfrac{1}{s}\right) = a_n \cdot s^{-n} + a_{n-1} \cdot s^{-n+1} + \cdots + a_1 s^{-1} + a_0$$

$$= s^{-n}\left(a_n + a_{n-1} \cdot s + \cdots + a_1 \cdot s^{n-1} + a_0 \cdot s^n\right)$$

$$\Rightarrow \hat{D}(s) = \frac{1}{s^n}\left(a_0 \cdot s^n + a_1 \cdot s^{n-1} + \cdots + a_{n-1} \cdot s + a_n\right)$$

$\Rightarrow$ Replacing $s \to \frac{1}{s}$ has the effect of reversing the order of the coefficients.

Example:

$$D(s) = s^5 + 2s^4 + 4s^3 + 8s^2 + 3s + 2$$

$$\Rightarrow \hat{D}(s) = 2s^5 + 3s^4 + 8s^3 + 4s^3 + 2s + 1$$

s^5	2	8	2
s^4	3	4	1
s^3	$5\frac{1}{3}$	$1\frac{1}{3}$	
s^2	$3\frac{1}{4}$	1	
s^1	$-\frac{4}{13}$		
s^0	1		

$\Rightarrow$ 2 sign changes $\Rightarrow$ 2 poles in RHP.
There was no need to work with ε.

Example:

$$D(s) = s^3 + 2s^2 + s + 2$$

$$\Rightarrow \hat{D}(s) = 2s^3 + s^2 + 2s + 1$$

s^3	2	2
s^2	1	1
s^1	ε	
s^0	1	

$\Rightarrow$ The problem remained the same, i.e., the ϕ would still show up. Of course, the answer is the same:

$\varepsilon \to \phi+$: $(+ + + +) \Rightarrow \phi$ sign changes

$\varepsilon \to \phi-$: $(+ + - +) \Rightarrow 2$ sign changes

$\Rightarrow$ 2 poles on imaginary axis.

<u>Example</u>:

$$D(s) = s^5 + 2s^4 + 5s^3 + 10s^2 + 4s + 8$$

$$\begin{array}{c|ccc}
s^5 & 1 & 5 & 4 \\
s^4 & 2 & 10 & 8 \\
s^3 & 0 & 0 & \\
 & & &
\end{array} \quad \Longleftarrow \text{ entire row } \equiv 0$$

If an entire row disappears, the row above represents a true divider of the original polynomial:

$$H(s) = 2s^4 + 10s^2 + 8$$

$$(s^5 + 2s^4 + 5s^3 + 10s^2 + 4s + 8) : (2s^4 + 10s^2 + 8) = \tfrac{1}{2}s + 1$$

$$\begin{array}{l}
s^5 \quad + 5s^3 \qquad + 4s \\
\hline
\diagdown \quad 2s^4 + \diagdown + 10s^2 + \diagdown + 8 \\
- \quad 2s^4 \qquad + 10s^2 \qquad + 8 \\
\hline
\quad \diagdown\!\diagdown \qquad\quad \diagdown\!\diagdown \qquad \diagdown
\end{array}$$

$$\Rightarrow D(s) = (s^4 + 5s^2 + 4) \cdot (s + 2)$$

In this case, we can actually find all the poles:

Substitution: $x = s^2$

$\Rightarrow x^2 + 5x + 4 = 0$

$\Rightarrow x_{1,2} = \dfrac{-5 \pm \sqrt{25-16}}{2} = \dfrac{-5 \pm 3}{2}$

$\Rightarrow x_1 = -1$
$x_2 = -4$

$s = \sqrt{x} \Rightarrow$
$$\left.\begin{aligned} s_1 &= j \\ s_2 &= -j \\ s_3 &= 2j \\ s_4 &= -2j \end{aligned}\right\}$$ 4 poles on imaginary axis.

$s_5 = -2$