Different Forms of Fourier Transform:

Our textbook represents the Fourier transform in terms of f :

$$\boxed{\begin{aligned} X(f) &= \int_{-\infty}^{+\infty} x(t)\cdot e^{-j2\pi ft}\,dt \\ x(t) &= \int_{-\infty}^{+\infty} X(f)\cdot e^{+j2\pi ft}\,df \end{aligned}}$$

The majority of the textbooks prefers to represent the Fourier transform in terms of $\omega = 2\pi f$:

$$\omega = 2\pi f$$
$$\Rightarrow d\omega = 2\pi \cdot df$$
$$\Rightarrow df = \frac{1}{2\pi}\cdot d\omega$$

f	ω
$-\infty$	$-\infty$
$+\infty$	$+\infty$

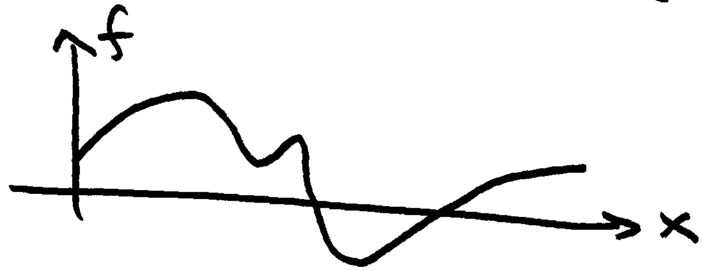
The Fourier transform still looks the same, but the inverse Fourier transform doesn't :

$$X(\omega) = \int_{-\infty}^{+\infty} x(t) \cdot e^{-j\omega t} \, dt$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\omega) \cdot e^{+j\omega t} \, d\omega$$

This is still the <u>same</u> Fourier transform, just represented in a different independent variable in the frequency domain.

However, this has consequences. Remember time-scaling:



$g(x) = f(2x)$

A function evaluated for $\xi = a \cdot x$ has the same form as the same function evaluated for x, but it is squeezed by a factor of a.

What is $\delta(at)$? It looks like $\delta(t)$, but it is squeezed by a factor of a. Therefore:

$$\int_{-\infty}^{+\infty} \delta(at)\,dt = \frac{1}{|a|}$$

$$\Rightarrow \boxed{\delta(at) = \frac{1}{|a|} \cdot \delta(t)}$$

$$\Rightarrow \delta(\omega) = \delta(2\pi f) = \frac{1}{2\pi} \cdot \delta(f)$$

$$\boxed{\begin{aligned} \delta(\omega) &= \frac{1}{2\pi} \cdot \delta(f) \\ \delta(f) &= 2\pi \cdot \delta(\omega) \end{aligned}}$$

$$\varepsilon(t) \circ\!\!-\!\!\bullet \frac{1}{2}\delta(f) + \frac{1}{j2\pi f}$$

$$\Rightarrow \boxed{\varepsilon(t) \circ\!\!-\!\!\bullet \pi \cdot \delta(\omega) + \frac{1}{j\omega}}$$

$$\mathcal{F}\left\{\int_{-\infty}^{t} x(\tau)\,d\tau\right\} = \mathcal{F}\{x(t)\} \cdot \mathcal{F}\{\varepsilon(t)\}$$

$$= X(\omega) \cdot \left[\pi \cdot \delta(\omega) + \frac{1}{j\omega}\right]$$

$$\Rightarrow \boxed{\mathcal{F}\left\{\int_{-\infty}^{t} x(\tau)\,d\tau\right\} = \frac{1}{j\omega} \cdot X(\omega) + \pi \cdot X(0) \cdot \delta(\omega)}$$

i.e., some formulae look slightly different when expressed in $\omega = 2\pi f$ instead of f.

The representation in ω has the <u>disadvantage</u> that the Fourier transform and inverse Fourier transform no longer look symmetrical because of the factor of $\frac{1}{2\pi}$. It has the advantage that it more naturally extends to the Laplace transform.

Some older books define the Fourier transform differently:

$$\boxed{\begin{aligned} X(\omega) &= \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{+\infty} x(t) \cdot e^{-j\omega t}\, dt \\ x(t) &= \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{+\infty} X(\omega) \cdot e^{+j\omega t}\, d\omega \end{aligned}}$$

This has the advantage of being symmetric again, but it is a differently scaled Fourier transform that no longer extends naturally

to the Laplace transform.
It is therefore <u>not</u> recommended.

Let us look at one more representation. We introduce the <u>complex radial frequency</u>:

$$\varpi := j\omega = j\,2\pi f$$

$$\Rightarrow d\varpi = j \cdot d\omega = 2\pi j \cdot df$$

$$\Rightarrow df = \frac{1}{2\pi j} \cdot d\varpi$$

f	ϖ
$-\infty$	$-j\infty$
$+\infty$	$+j\infty$

$$\Rightarrow \left| \begin{aligned} X(\varpi) &= \int_{-\infty}^{+\infty} x(t) \cdot e^{-\varpi t} \cdot dt \\ x(t) &= \frac{1}{2\pi j} \int_{-j\infty}^{+j\infty} X(\varpi) \cdot e^{+\varpi t} \cdot d\varpi \end{aligned} \right|$$