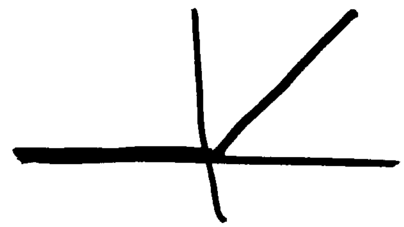
Generalized Fourier Transforms:

What is the Fourier transform of

$$x(t) = t \cdot \varepsilon(t)$$



$$X(f) = \int_{-\infty}^{+\infty} t \cdot \varepsilon(t) \cdot e^{-j2\pi ft}\, dt$$

does not exist!

$\Rightarrow$ Not all functions have Fourier transforms.

A necessary and sufficient condition for the existence of the Fourier transform is:

$$\int_{-\infty}^{+\infty} |x(t)|\, dt = M < \infty \qquad (1)$$

Unfortunately, condition (1) excludes a number of very useful functions, including

$$x(t) = \sin(t)$$

a periodic function with a Fourier series, but without a Fourier transform in the narrow sense.

The generalized Fourier transform extends the Fourier transform to include some useful functions, such as:

$$x(t) = \sin(t)$$
$$x(t) = \delta(t)$$
$$x(t) = A$$

It is important to generalize in a natural way, such that the generalizations are not in contradiction with the normal Fourier transforms, and such that they are consistent among themselves.

<u>Duality Theorem</u>:

$$\left|\begin{array}{l} x(t) = \displaystyle\int_{-\infty}^{+\infty} X(f)\cdot e^{+j2\pi ft}\,dt \\ X(f) = \displaystyle\int_{-\infty}^{+\infty} x(t)\cdot e^{-j2\pi ft}\,dt \end{array}\right|$$

These formulae are almost symmetric.

$$x(t) \circ\!\!-\!\!\bullet\, X(f)$$

$$\Updownarrow$$

$$X(t) \circ\!\!-\!\!\bullet\, x(-f)$$

i.e., if we know a Fourier transform pair $x(t), X(f)$, we can immediately obtain a second one by exchanging the symbols t and f, and replacing f by -f.

Example:

$$x(t) = \Pi(t) \circ\!\!-\!\!\bullet\; X(f) = \mathrm{sinc}(f)$$

$$\Updownarrow$$

$$x(t) = \mathrm{sinc}(t) \circ\!\!-\!\!\bullet\; X(f) = \Pi(-f) \equiv \Pi(f)$$

Let us find the Fourier transform of $\delta(t)$:

$$\mathcal{F}\{\delta(t)\} = \int_{-\infty}^{+\infty} \delta(t) \cdot e^{-j2\pi f t}\, dt$$

$$= e^{-j2\pi f \emptyset} = 1$$

$$\boxed{\delta(t) \circ\!\!-\!\!\bullet\; 1}$$

Unfortunately, the Fourier transform of 1 does not exist.

$\Rightarrow$ We extend the definition by ensuring compliance with the duality theorem:

$$\delta(t) \circ\!\!-\!\!\bullet\ 1$$

$$\Updownarrow$$

$$1 \circ\!\!-\!\!\bullet\ \delta(-f) = \delta(f)$$

$$\boxed{1 \circ\!\!-\!\!\bullet\ \delta(f)}$$

is a <u>generalized</u> Fourier transform.

Frequency Translation Theorem:

Given the pair:

$$x(t) \circ\!\!-\!\!\bullet X(f)$$

Question: What is the Fourier transform of $x(t) \cdot e^{j\omega_0 t}$?

$$\int_{-\infty}^{+\infty} x(t) \cdot e^{j\omega_0 t} \cdot e^{-j2\pi f t}\, dt$$

$$= \int_{-\infty}^{+\infty} x(t) \cdot e^{j2\pi f_0 t} \cdot e^{-j2\pi f t}\, dt$$

$$= \int_{-\infty}^{+\infty} x(t) \cdot e^{-j2\pi(f-f_0)t}\, dt$$

$$\Rightarrow \boxed{x(t) \cdot e^{j\omega_0 t} \circ\!\!-\!\!\bullet X(f-f_0)}$$

Modulation Theorem:

Question: What is the Fourier transform of $x(t)\cdot\cos(\omega_0 t)$?

$$x(t)\cdot\cos(\omega_0 t) = x(t)\cdot\frac{e^{j\omega_0 t}+e^{-j\omega_0 t}}{2}$$

$$= \frac{1}{2}x(t)\cdot e^{j\omega_0 t} + \frac{1}{2}x(t)\cdot e^{-j\omega_0 t}$$

$$\Rightarrow \boxed{x(t)\cdot\cos(\omega_0 t) \circ\!\!-\!\!\bullet \frac{1}{2}X(f-f_0)+\frac{1}{2}X(f+f_0)}$$

Similarly:

$$x(t)\cdot\sin(\omega_0 t) = x(t)\cdot\frac{e^{j\omega_0 t}-e^{-j\omega_0 t}}{2j}$$

$$\Rightarrow \boxed{x(t)\cdot\sin(\omega_0 t) \circ\!\!-\!\!\bullet \frac{1}{2j}X(f-f_0)-\frac{1}{2j}X(f+f_0)}$$

Unfortunately, the Fourier transform of $\cos(\omega_0 t)$ does not exist. We extend the definition by applying the modulation theorem to the generalized Fourier transform of 1:

$$\cos(\omega_0 t) = 1 \cdot \cos(\omega_0 t)$$

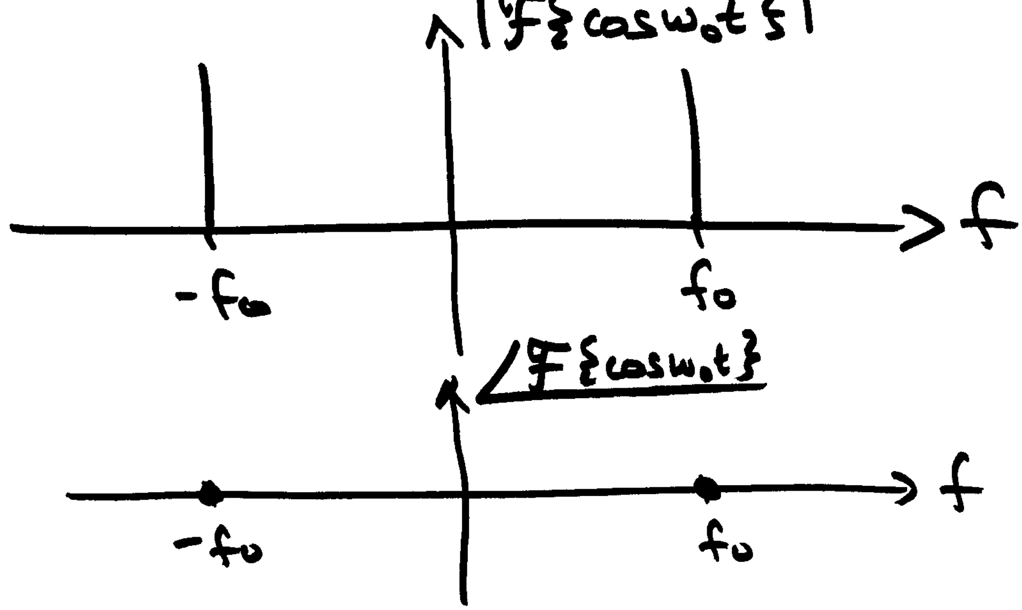
$$\Rightarrow \boxed{\cos(\omega_0 t) \circ\!\!-\!\!\bullet \frac{1}{2} \cdot \delta(f - f_0) + \frac{1}{2} \cdot \delta(f + f_0)}$$

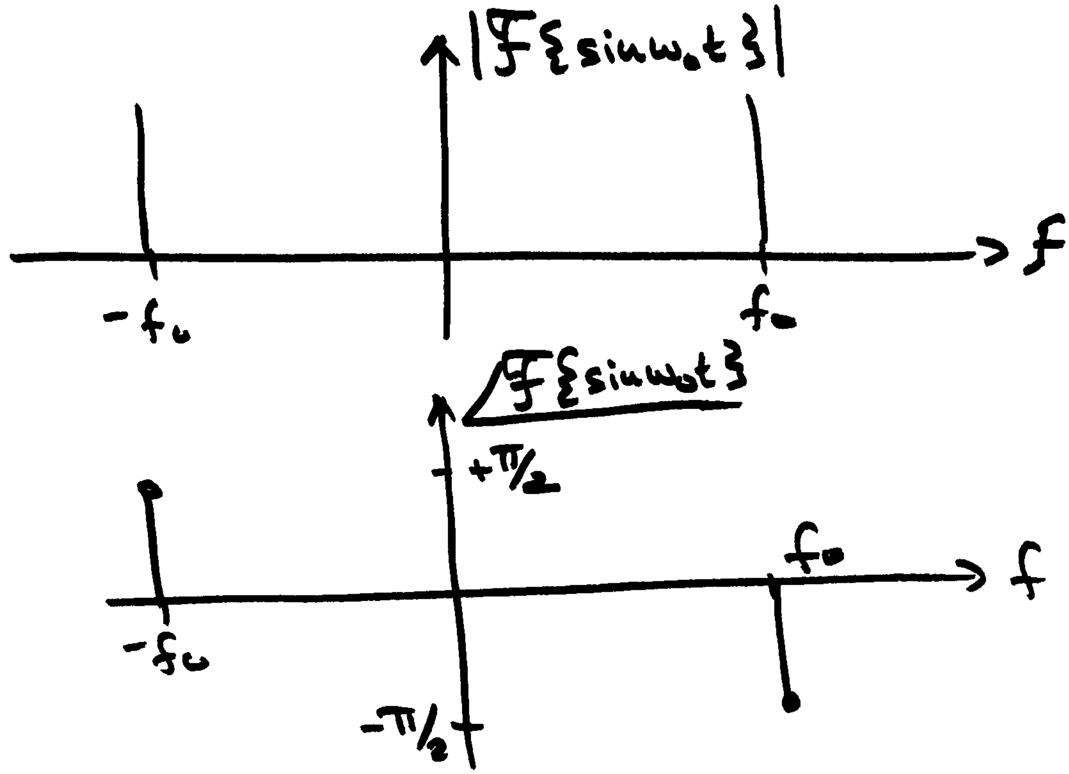
Similarly:

$$\boxed{\sin(\omega_0 t) \circ\!\!-\!\!\bullet \frac{1}{2j} \cdot \delta(f - f_0) - \frac{1}{2j} \cdot \delta(f + f_0)}$$

These are again <u>generalized</u> Fourier transforms.

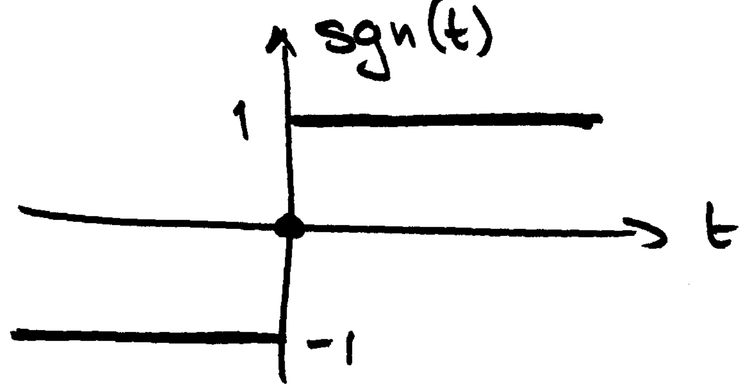
These extensions make sense:



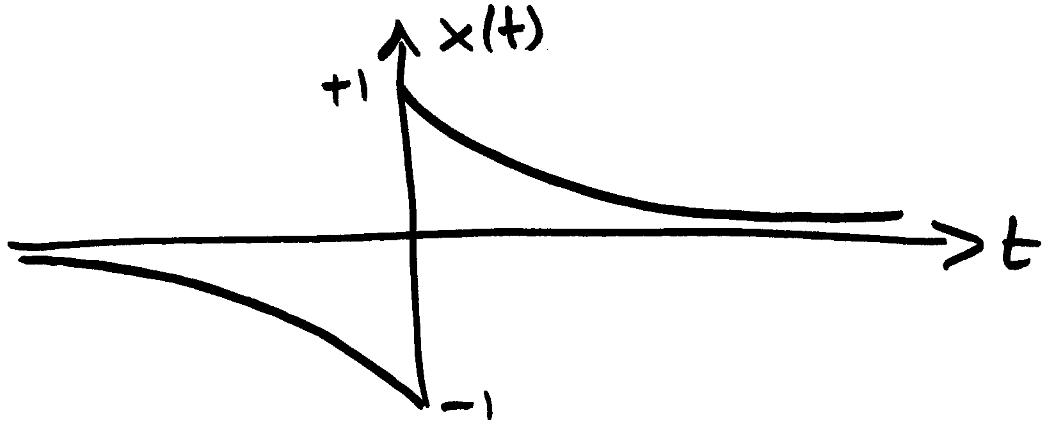


The generalized Fourier transforms of $\cos(\omega_0 t)$ and $\sin(\omega_0 t)$ are Dirac impulses at $\pm\omega_0$ $\frac{\text{rad}}{\text{sec}}$ or at $\pm f_0$ Hz. This is consistent with the Fourier series of the previous chapter.

What is the Fourier transform of the signum function?



Unfortunately, also this Fourier transform does not exist. It has become customary to use approximation:



$$x(t) = e^{-\eta t} \cdot \varepsilon(t) - e^{+\eta t} \cdot \varepsilon(-t)$$

$x(t)$ **does** have a Fourier transform:

$$\mathcal{F}\{x(t)\} = \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi f t}\, dt$$

$$= \int_{-\infty}^{+\infty} \left(e^{-\eta t} \cdot \varepsilon(t) - e^{+\eta t} \cdot \varepsilon(-t) \right) \cdot e^{-j2\pi f t}\, dt$$

$$= -\int_{-\infty}^{0} e^{+\eta t} \cdot e^{-j2\pi f t}\, dt + \int_{0}^{\infty} e^{-\eta t} \cdot e^{-j2\pi f t}\, dt$$

$$= \frac{-1}{\eta - j2\pi f}\left[e^{\eta t}\cdot e^{-j2\pi f t}\right]_{-\infty}^{0}$$

$$+ \frac{1}{-\eta - j2\pi f}\left[e^{-\eta t}\cdot e^{-j2\pi f t}\right]_{0}^{\infty}$$

$$= \frac{-1}{\eta - j2\pi f} - \frac{1}{-\eta - j2\pi f}$$

$$\Rightarrow \mathcal{F}\{x(t)\} = \frac{-1}{\eta - j2\pi f} - \frac{1}{-\eta - j2\pi f} \; ; \; \underline{\underline{\forall \eta > \emptyset}}$$

$$\operatorname{sgn}(t) = \lim_{\eta \to 0} x(t) = \lim_{\eta \to 0}\left\{e^{-\eta t}\cdot \varepsilon(t) - e^{+\eta t}\cdot \varepsilon(-t)\right\}$$

$$\mathcal{F}\{\operatorname{sgn}(t)\} = \mathcal{F}\left\{\lim_{\eta \to 0} x(t)\right\}$$

does not exist.

We extend the definition by defining:

$$\mathcal{F}\{\operatorname{sgn}(t)\} = \lim_{\eta \to 0}\left\{\mathcal{F}\{x(t)\}\right\}$$

which does exist.

$$\Rightarrow \mathcal{F}\{\mathrm{sgn}(t)\} = \lim_{\eta \to 0} \left\{ \frac{-1}{\eta - j2\pi f} - \frac{1}{-\eta - j2\pi f} \right\}$$

$$= \frac{+2}{j2\pi f} = +\frac{1}{j\pi f}$$

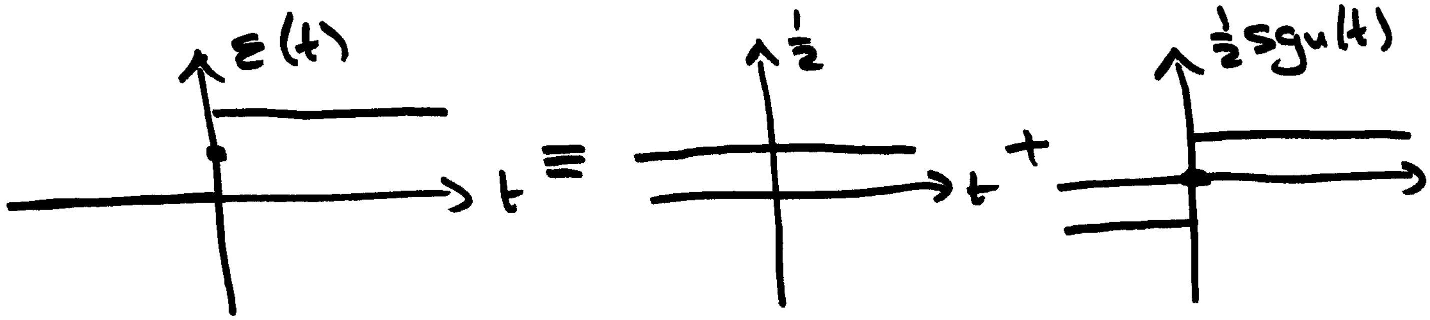
$$\boxed{\mathrm{sgn}(t) \circ\!\!-\!\!\bullet \frac{1}{j\pi f}}$$

is a <u>generalized</u> Fourier transform.

These are all the extensions that we shall make. Especially the last one (interchanging the $\mathcal{F}\{\cdot\}$ and $\lim\{\cdot\}$ operators) is <u>GOOFY</u>, as it easily leads to inconsistent results.

What is the Fourier transform of $\varepsilon(t)$? Unfortunately, it does not exist. However, we can write:

$$\varepsilon(t) = \frac{1}{2} + \frac{1}{2}\,\mathrm{sgn}(t)$$



$$\Rightarrow \boxed{\varepsilon(t) \;\circ\!\!-\!\!\bullet\; \frac{1}{2}\,\delta(f) + \frac{1}{j2\pi f}}$$

However:

$$\varepsilon(t) = \lim_{\eta \to 0} e^{-\eta t} \cdot \varepsilon(t) \quad ; \forall \eta > 0$$

is correct

$$\mathcal{F}\{\varepsilon(t)\} = \mathcal{F}\{\lim_{\eta \to 0} e^{-\eta t} \cdot \varepsilon(t)\}$$

does not exist.

↓ Generalization ?

$$\lim_{\eta \to 0} \{\mathcal{F}\{e^{-\eta t} \cdot \varepsilon(t)\}\}$$

$$= \lim_{\eta \to 0} \left\{ \int_0^\infty e^{-\eta t} \cdot e^{-j2\pi f t} dt \right\}$$

$$= \lim_{\eta \to 0} \left\{ \frac{1}{-\eta - j2\pi f} \left[e^{-\eta t} \cdot e^{-j2\pi f t} \right]_0^\infty \right\}$$

$$= \lim_{\eta \to 0} \left\{ \frac{1}{\eta + j2\pi f} \right\} = \frac{1}{j2\pi f}$$

IS INCONSISTENT!!

The $\mathcal{F}\{\cdot\}$ and $\lim\{\cdot\}$ operators cannot be interchanged arbitrarily without leading to inconsistent answers!

Fourier transform <u>without</u> generalizations is mathematically sound, but not useful.

Fourier transform <u>with</u> generalizations is useful, but not mathematically sound!
