Stability:

Given a linear system:

$$\left| \begin{array}{l} \dot{\underline{x}} = A\underline{x} + \underline{b}u \\ y = \underline{c}'\underline{x} \end{array} \right| \qquad \underline{x}(t=\emptyset) = \underline{x}_0$$

$$\Rightarrow y(t) = \underline{c}' e^{At} \cdot \underline{x}_0 + \underline{c}' \int_{0-}^{t} e^{A(t-\tau)} \underline{b}\, u(\tau)\, d\tau$$

We want to analyze under what conditions remains $y(t) < \infty$ for $u(t) < \infty$, $\forall \underline{x}_0$, $\forall t > \emptyset$.

We do this in two parts:

→ input response $(u = u(t),\ \underline{x}_0 = \emptyset)$
→ state response $(u = \emptyset,\ \underline{x}_0 = \underline{x}_0)$

using the superposition principle.

A) Input response:

$$y(t) = \underline{c}' \int_{0-}^{t} e^{A(t-\tau)} \underline{b}\, u(\tau)\, d\tau$$

$$\equiv \int_{0-}^{t} g(t-\tau)\, u(\tau)\, d\tau$$

$$\equiv g(t) * u(t)$$

$$\Rightarrow |y(t)| = \left| \int_{0-}^{t} g(t-\tau)\cdot u(\tau)\,d\tau \right|$$

$$\leq \int_{0-}^{t} |g(t-\tau)\cdot u(\tau)|\,d\tau$$

$$= \int_{0-}^{t} |g(t-\tau)|\cdot|u(\tau)|\,d\tau$$

If the input is bounded:

$$|u(t)| \leq M < \infty$$

$$\Rightarrow |y(t)| \leq \int_{0-}^{t} |g(t-\tau)|\cdot|u(\tau)|\,d\tau$$

$$\leq \int_{0-}^{t} |g(t-\tau)|\cdot M\,d\tau$$

$$= M\cdot\int_{0-}^{t} |g(t-\tau)|\,d\tau$$

If $|y(t)| \leq N < \infty$

$$\Longrightarrow \int_{0-}^{t} |g(t-\tau)|\,d\tau \leq \frac{N}{M} < \infty$$

is <u>sufficient</u> for stability of input response, the so-called Bounded-Input / Bounded-Output (BIBO) stability.

It is also a <u>necessary</u> condition, because this must work for <u>all</u> $u(t)$, in particular for:

$$u(t) = \begin{cases} +1 \; ; & g(t-\tau) > \phi \\ \phi \; ; & g(t-\tau) = \phi \\ -1 \; ; & g(t-\tau) < \phi \end{cases}$$

In this special case:

$$|y(t)| = \left| \int_{0-}^{t} g(t-\tau) \cdot u(\tau) d\tau \right| \equiv \left| \int_{0-}^{t} |g(t-\tau)| d\tau \right|$$

$$\equiv \int_{0-}^{t} |g(t-\tau)| d\tau$$

q.e.d.

Let us introduce the variable transformation:

$$\vartheta = t - \tau$$
$$d\vartheta = -d\tau$$

τ	ϑ
$0-$	t
t	$0-$

$$\Rightarrow \int_{0-}^{t} |g(t-\tau)|\, d\tau \equiv -\int_{t}^{0-} |g(\vartheta)|\, d\vartheta$$

$$\equiv \int_{0-}^{t} |g(\vartheta)|\, d\vartheta$$

Hence a necessary and sufficient condition for BIBO stability is:

$$\int_{0-}^{t} |g(\tau)|\, d\tau < \infty$$

$$g(t) = \mathcal{L}^{-1}\{G(s)\}$$

$$G(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$

Partial fraction expansion:

$$G(s) = \frac{r_1}{s-p_1} + \frac{r_2}{s-p_2} + \cdots + \frac{r_n}{s-p_n}$$

The poles, p_i, are the roots of $D(s)$. They are not influenced by $N(s)$. The residua, r_i, are determined by $N(s)$.

If there are multiple poles, the PFE may have terms such as:

$$\frac{r_{ik}}{(s-p_i)^k} \quad .$$

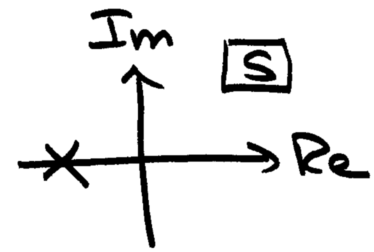
$$\Rightarrow g(t) = \mathcal{L}^{-1}\{G(s)\} = r_1 e^{p_1 t} + r_2 \cdot e^{p_2 t} + \cdots + r_n e^{p_n t}$$

Multiple poles would produce terms such as:

$$r_{ik} \cdot t^k \cdot e^{p_i t}$$

Different cases:

(a) p_i negative and real:



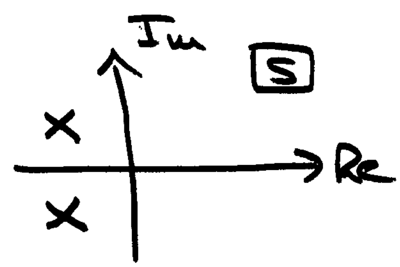
$\Rightarrow r_i e^{p_i t}$

$r_{ik} t^k e^{p_i t}$

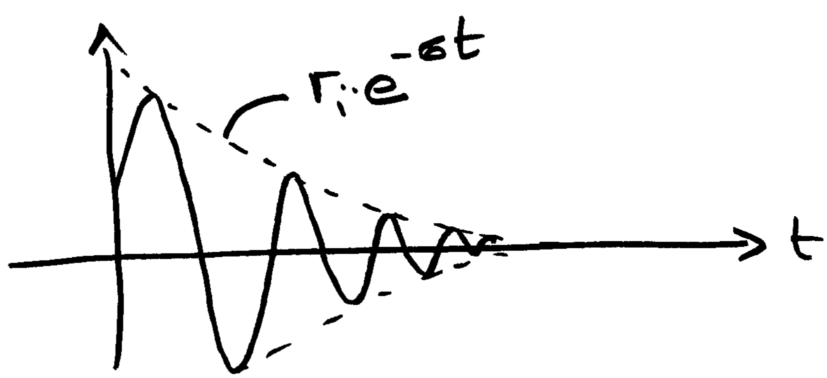
$\Rightarrow$ remain bounded.

(b) p_i negative and complex (always in pairs):

$$p_i = -\sigma \pm j\omega$$



$$\Rightarrow r_i e^{p_i t} = r_i e^{-\sigma t} \cdot e^{\pm j\omega t}$$
$$= r_i e^{-\sigma t} \cdot (\cos \omega t \pm j \sin \omega t)$$



remains bounded.

Multiplication with t^k does not significantly change things.

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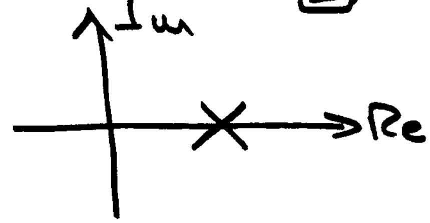
(c) p_i positive and real:

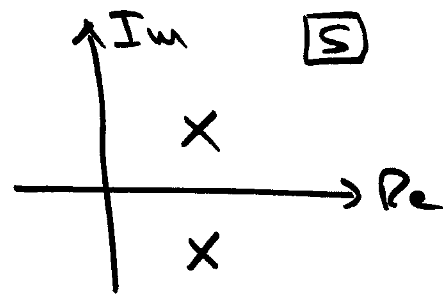


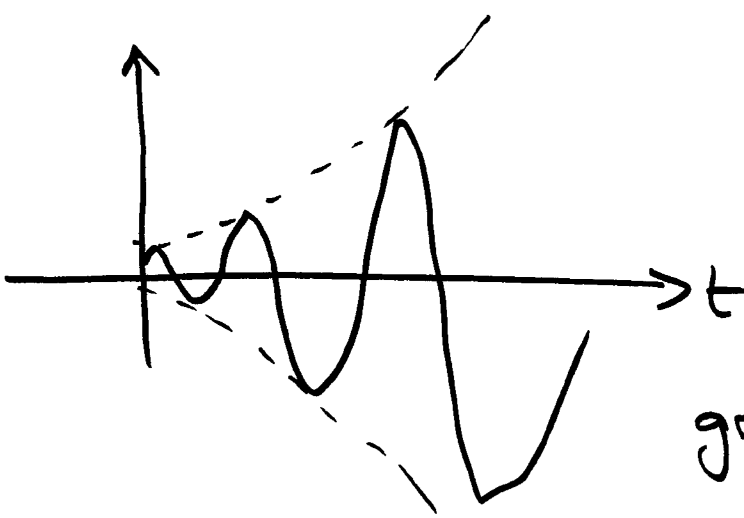
$\Rightarrow r_i e^{p_i t}$

grows beyond all bounds

$\boxed{5}$

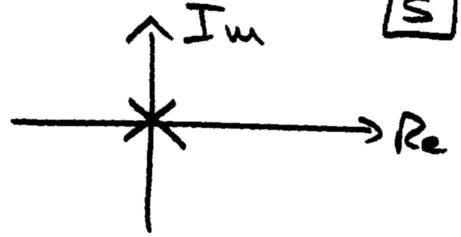
(d) p_i positive and complex:





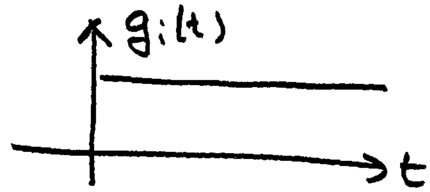
grows beyond all bounds

(e) P_i at origin:



$$\Rightarrow G_i(s) = \frac{r_i}{s}$$

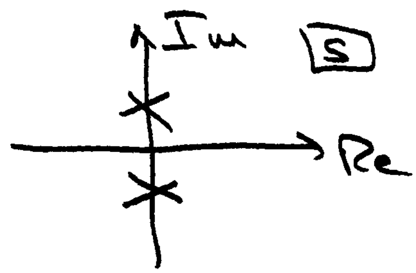
$$\Rightarrow g_i(t) = r_i \cdot \varepsilon(t)$$



$$\Rightarrow \int_0^t |g_i(\tau)|\, d\tau$$

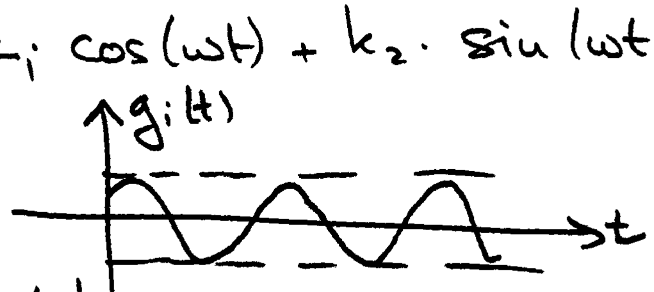
grows beyond all bounds as $t \to \infty$.

(f) P_i on imaginary axis:

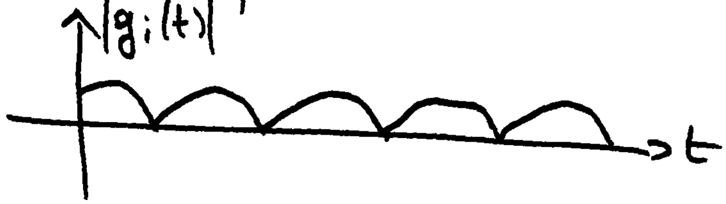


$$\Rightarrow G_i(s) = \frac{r_{i1}}{s+j\omega} + \frac{r_{i2}}{s-j\omega} = \frac{k}{s^2+\omega^2}$$

$$\Rightarrow g_i(t) = k_1 \cos(\omega t) + k_2 \cdot \sin(\omega t)$$

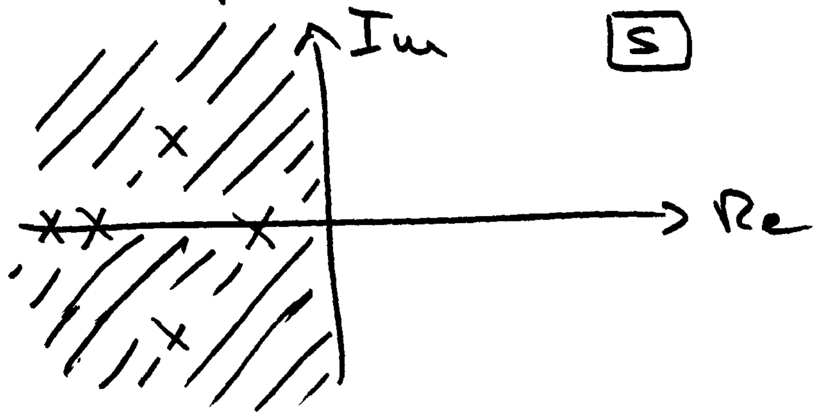


$\Rightarrow |g_i(t)| =$



$$\Rightarrow \int_{0^-}^{t} |g_i(\tau)|\,d\tau \quad \text{grows beyond all bounds as } t \to \infty.$$

<u>Thus</u>: a necessary <u>and</u> sufficient condition for BIBO stability is that all poles of $G(s)$ are in the <u>open</u> left half plane (LHP):



$$\Rightarrow \max(\text{real}(\text{roots}(D(s)))) < 0$$

In Matlab:

```
sys=ss(A,b,c,d)
G= tf(sys)
[N,D]= tfdata(G,'v')
r= roots(D)
rr= real(r)
rmax = max(rr)
```

If rmax < 0 $\Rightarrow$ system is BIBO stable.

$$G(s) = \underline{c}' \cdot (sI-A)^{-1} \cdot \underline{b}$$

$$\equiv \frac{\underline{c}' \cdot (sI-A)^{+} \cdot \underline{b}}{|sI-A|}$$

$$\Rightarrow N(s) = \underline{c}' \cdot (sI-A)^{+} \cdot \underline{b}$$

$$D(s) = |sI-A|$$

The roots of $D(s)$ are the roots of $|sI - A|$, i.e., the roots of the <u>characteristic polynomial</u> of the system.

These are also the <u>eigenvalues</u> of the matrix A (see later!).

<u>In Matlab</u>:

```
l = eig(A)
rl = real(l)
lmax = max(rl)
```

If $lmax < \emptyset \implies$ system is BIBO stable.

<u>Warning</u>: The results may not be the same because of possible pole/zero cancellations of $G(s)$.

$$G(s) = \frac{\cdots \cancel{(p-3)} \cdots}{\cdots \cancel{(p-3)} \cdots}$$

no longer shows up!

B) <u>State Response</u>:

$$y(t) = \underline{c}' \cdot e^{At} \cdot \underline{x}_0$$

$$\circ\!\!-\!\!\bullet$$

$$Y(s) = \underline{c}' \cdot (sI - A)^{-1} \cdot \underline{x}_0$$

$$\equiv \frac{\underline{c}' \cdot (sI-A)^{+} \cdot \underline{x}_0}{|sI - A|}$$

Since stability is only influenced by the denominator, the results are the same.

Stability to the initial conditions is called <u>Lyapunov</u> stability. (The spelling varies with the text book you use. Correct spelling: Ляпунов = ЛЯПУНОВ)

However, since this must be true for **all** $\underline{x}_0$, there is no longer any benefit possible of pole/zero cancellation.

$\Rightarrow$ Test of eigenvalues is better!