Numerical Simulation of Dynamic Systems IX

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Given a mechanical system:

\[ k_1 = 0.3 \text{ kg sec}^{-2} \]
\[ B_1 = 0.8 \text{ kg sec}^{-1} \]
\[ k_2 = 8 \text{ kg sec}^{-2} \]
\[ B_2 = 10 \text{ kg sec}^{-1} \]
\[ k_3 = 3 \text{ kg sec}^{-2} \]
\[ B_3 = 12 \text{ kg sec}^{-1} \]

**Figure:** Mechanical model of a sitting human body
We can obtain a set of differential equations using Newton’s law:

\[
\begin{align*}
M_1 \ddot{x}_1 &= k_1 \cdot (x_2 - x_1) + B_1 \cdot (\dot{x}_2 - \dot{x}_1) \\
M_2 \ddot{x}_2 &= k_2 \cdot (x_3 - x_2) + B_2 \cdot (\dot{x}_3 - \dot{x}_2) + k_3 \cdot (x_4 - x_2) \\
&\quad + B_3 \cdot (\dot{x}_4 - \dot{x}_2) - k_1 \cdot (x_2 - x_1) - B_1 \cdot (\dot{x}_2 - \dot{x}_1) \\
M_3 \ddot{x}_3 &= -k_2 \cdot (x_3 - x_2) - B_2 \cdot (\dot{x}_3 - \dot{x}_2) \\
M_4 \ddot{x}_4 &= F - k_3 \cdot (x_4 - x_2) - B_3 \cdot (\dot{x}_4 - \dot{x}_2)
\end{align*}
\]
Introduction II

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Typically, DAE models derived from mechanical systems contain second derivatives.

In general, we can obtain models of the form:

\[
\dot{x} = f(x, \dot{x}, u, t)
\]
Although it is always possible to convert such models to state-space form by augmenting the state vector by the velocity vector $\mathbf{v} = \dot{x}$:

$$
\begin{align*}
\dot{x} & = \mathbf{v} \\
\dot{\mathbf{v}} & = f(x, \mathbf{v}, \mathbf{u}, t)
\end{align*}
$$

this may not necessarily be desirable.
Introduction III

Although it is always possible to convert such models to state-space form by augmenting the state vector by the velocity vector $v = \dot{x}$:

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this may not necessarily be desirable.

It may be worthwhile to investigate whether we could find numerical ODE solvers that can deal with second-derivative models directly. This is the purpose of today’s presentation.
Introduction IV

We can reformulate the human body model in a matrix vector form using the *partial state vector* \( \mathbf{x} = (x_1, x_2, x_3, x_4)^T \):

\[
\mathbf{M} \cdot \ddot{\mathbf{x}} + \mathbf{C} \cdot \dot{\mathbf{x}} + \mathbf{K} \cdot \mathbf{x} = \mathbf{f}
\]

where:

\[
\mathbf{M} = \begin{pmatrix}
M_1 & 0 & 0 & 0 \\
0 & M_2 & 0 & 0 \\
0 & 0 & M_3 & 0 \\
0 & 0 & 0 & M_4
\end{pmatrix};
\quad
\mathbf{C} = \begin{pmatrix}
B_1 & -B_1 & 0 & 0 \\
-B_1 & (B_1 + B_2 + B_3) & 0 & 0 \\
0 & -B_2 & B_2 & 0 \\
0 & -B_3 & 0 & B_3
\end{pmatrix}
\]

\[
\mathbf{K} = \begin{pmatrix}
k_1 & -k_1 & 0 & 0 \\
-k_1 & (k_1 + k_2 + k_3) & 0 & 0 \\
0 & -k_2 & k_2 & 0 \\
0 & -k_3 & 0 & k_3
\end{pmatrix};
\quad
\mathbf{f} = \begin{pmatrix}
0 \\
0 \\
0 \\
F
\end{pmatrix}
\]

\( \mathbf{M} \) is the *mass matrix*, \( \mathbf{C} \) is the *damping matrix*, \( \mathbf{K} \) is the *stiffness matrix*, and \( \mathbf{f} \) is the *vector of (generalized) forces*. 
The mass matrix turned out to be a diagonal matrix in this example, but this is only true, because no rotational motions were considered in the given example. Generally, this will not be the case.

Assuming that the mass matrix is non-singular, i.e., there are as many mechanical degrees of freedom in the system as were formulated into second-order differential equations, i.e., there are no structural singularities in the model, the model can be solved for the highest derivatives:

\[ \ddot{x} = A^2 \cdot x + B \cdot \dot{x} + u \]

where:

\[ A = \sqrt{-M^{-1} \cdot K} \]
\[ B = -M^{-1} \cdot C \]
\[ u = M^{-1} \cdot f \]
Of special interest is the case of the \textit{conservative (i.e., friction-less) systems} with the second-derivative form:

\[ \ddot{x} = A^2 \cdot x + u \]

and especially, we may want to look at homogeneous, conservative, linear systems with the second-derivative model:

\[ \ddot{x} = A^2 \cdot x \]
Velocity-free Models

We shall define a *velocity-free model* as one that satisfies, in the linear case, the differential vector equation:

\[ \ddot{x} = A^2 \cdot x + u \]

and, in the non-linear case, the differential vector equation:

\[ \ddot{x} = f(x, u, t) \]
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Every conservative system leads to a velocity-free second-derivative model. Yet, not every velocity-free second-derivative model is conservative.
Given a linear, time-invariant, homogeneous state-space model of the form:

\[ \dot{x} = A \cdot x \]

Depending on the eigenvalues of \( A \), the system is either damped or undamped, stable or unstable.
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Any linear, time-invariant, homogeneous state-space model can also be written in the form of a velocity-free second-derivative model, irrespective of where its eigenvalues are located. Yet, a conservative linear system has its eigenvalues spread up and down along the imaginary axis of the complex plane.
Velocity-free Models III

How can the special structure of a velocity-free second-derivative model be exploited by a simulation algorithm?
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We start by developing the solution vector at time $(t + h)$ into a Taylor series around time $t$:

$$x_{k+1} = x_k + h \cdot \dot{x}_k + \frac{h^2}{2} \cdot \ddot{x}_k + \frac{h^3}{6} \cdot x_k^{(iii)} + \frac{h^4}{24} \cdot x_k^{(iv)} + \ldots$$

We also need to develop the solution vector at time $(t - h)$ into a Taylor series around time $t$:

$$x_{k-1} = x_k - h \cdot \dot{x}_k + \frac{h^2}{2} \cdot \ddot{x}_k - \frac{h^3}{6} \cdot x_k^{(iii)} + \frac{h^4}{24} \cdot x_k^{(iv)} + \ldots$$
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\]

Adding these two equations together, we obtain:

\[
x_{k+1} + x_{k-1} = 2 \cdot x_k + h^2 \cdot \ddot{x}_k + \frac{h^4}{12} \cdot x_k^{(iv)} + \ldots
\]
Truncated after the quadratic term:

\[ x_{k+1} = 2 \cdot x_k - x_{k-1} + h^2 \cdot \ddot{x}_k \]
Velocity-free Models IV

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We just found a $3^{rd}$-order accurate explicit linear multi-step method that makes use of the second derivative directly. In some references, the method is referred to as Godunov's method (GE3).
Velocity-free Models IV

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We just found a 3\textsuperscript{rd}-order accurate explicit linear multi-step method that makes use of the second derivative directly. In some references, the method is referred to as Godunov’s method (GE3).

Plugging in the homogeneous linear second-derivative model:

\[ x_{k+1} = 2 \cdot x_k - x_{k-1} + (A \cdot h)^2 \cdot x_k \]
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Plugging in the homogeneous linear second-derivative model:

\[ x_{k+1} = 2 \cdot x_k - x_{k-1} + (A \cdot h)^2 \cdot x_k \]

Let:

\[ \xi_k = \begin{pmatrix} x_{k-1} \\ x_k \end{pmatrix} \]

Then:

\[ \xi_{k+1} \approx F \cdot \xi_k \quad ; \quad F = \begin{pmatrix} Z^{(n)} & I^{(n)} \\ -I^{(n)} & 2 \cdot I^{(n)} + (A \cdot h)^2 \end{pmatrix} \]
Stability and Damping of GE3

Before we attempt to draw a stability domain of GE3, we shall draw the linear damping plot:

Figure: Linear damping plot of GE3 algorithm
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![Linear Damping Plot of GE3](image)

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How very disappointing! The scheme is unstable in the left half plane!
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![Linear Damping Plot of GE3](image)

**Figure:** Linear damping plot of GE3 algorithm

How very disappointing! The scheme is unstable in the left half plane!

The result should not surprise us too much. Since the $F$-matrix is an *even function* in $A \cdot h$, the damping properties must be symmetric to the imaginary axis. Thus there cannot exist an asymptotic region around the origin, as we would expect of any well-behaved integration algorithm.
To gain a better understanding of the damping properties of the algorithm, let us plot the damping order star.
Interesting is the line segment stretching from \(-2j\) to \(+2j\) along the imaginary axis. Evidently, there is zero damping along this line segment, which is exactly, what it should be. To verify the results, let us plot the linear damping properties once more, but this time along the imaginary rather than the real axis.

**Figure**: Linear damping properties of GE3 along imaginary axis
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In order to obtain marginally stable results, the largest absolute eigenvalue multiplied by the step size must be smaller than or equal to 2:

$$|\lambda|_{\text{max}} \cdot h \leq 2$$
Let us now plot the linear frequency properties of GE3 along the imaginary axis.

**Figure:** Linear frequency plot of GE3
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Figure: Linear frequency plot of GE3

The algorithm produces results that are decently accurate for:

$$|\lambda_{\text{max}}\cdot h| \leq 1$$
Stability and Damping of GE3 V

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▶ The algorithm is therefore only useful for the simulation of linear conservation laws, i.e., either linear mechanical systems that are strictly conservative or linear hyperbolic partial differential equations (PDEs), i.e., the wave equation. We shall deal with the simulation of distributed parameter systems in the next chapter.
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- These results are unfortunately not very exiting, because the restrictions are too severe. How often will it happen that I wish to simulate a pure linear wave equation or a strictly conservative linear mechanical system?
Linear Velocity Models

Given:

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We apply the following variable transformation:

\[ \xi = \exp \left( \frac{B \cdot t}{2} \right) \cdot x \]

Therefore:

\[ x = \exp \left( \frac{-B \cdot t}{2} \right) \cdot \xi \]
\[ \dot{x} = -\frac{B}{2} \cdot \exp \left( \frac{-B \cdot t}{2} \right) \cdot \xi + \exp \left( \frac{-B \cdot t}{2} \right) \cdot \dot{\xi} \]
\[ \ddot{x} = \frac{B^2}{4} \cdot \exp \left( \frac{-B \cdot t}{2} \right) \cdot \xi - B \cdot \exp \left( \frac{-B \cdot t}{2} \right) \cdot \dot{\xi} + \exp \left( \frac{-B \cdot t}{2} \right) \cdot \ddot{\xi} \]
With the abbreviation:

$$E = \exp \left( -\frac{B \cdot t}{2} \right)$$

we find:

$$\begin{align*}
  x &= E \cdot \xi \\
  \dot{x} &= -\frac{B}{2} \cdot E \cdot \xi + E \cdot \dot{\xi} \\
  \ddot{x} &= \frac{B^2}{4} \cdot E \cdot \xi - B \cdot E \cdot \dot{\xi} + E \cdot \ddot{\xi}
\end{align*}$$
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\[ \ddot{x} = \frac{B^2}{4} \cdot E \cdot \dot{\xi} - B \cdot E \cdot \ddot{\xi} + E \cdot \dddot{\xi} \]

Plugging these expressions into the *linear-velocity second-derivative model*:

\[ \dddot{\xi} = E^{-1} \cdot \frac{B^2}{4} \cdot E \cdot \xi + E^{-1} \cdot f(E \cdot \xi, u, t) \]

we convert this model to a *velocity-free second-derivative model*. 
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- Although it is possible to convert any *linear velocity model* mathematically into an equivalent *velocity-free model*, the conversion is not helpful, because the resulting velocity-free model is not strictly conservative, i.e., does not have the eigenvalues of its Jacobian located on the imaginary axis at all times.
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- Yet, we cannot simulate arbitrary velocity-free models using the GE3 algorithm, but only the small sub-class of linear conservation laws.
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- Yet, we cannot simulate arbitrary velocity-free models using the GE3 algorithm, but only the small sub-class of *linear conservation laws*.

- Godunov was justified in his approach, because he was explicitly and exclusively interested in the simulation of linear conservation laws, and for this special problem, the GE3 algorithm is ideally suited.
Conclusions

In this presentation, we have started to look at a new class of linear multi-step algorithms, specifically designed for the simulation of second derivative systems, i.e., models that contain the second derivatives of the partial state vector explicitly in their model equations.
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- Unfortunately, the GE3 algorithm turned out to be a disappointment, because it cannot be used to simulate systems with damping.

- The method is numerically unstable everywhere in the open left-half complex plane, because its \( F \)-matrix turned out to be an even function in \( A \cdot h \).
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- The method is numerically unstable everywhere in the open left-half complex plane, because its F-matrix turned out to be an even function in $A \cdot h$.

- We’ll definitely need something better.