Numerical Simulation of Dynamic Systems XXV

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May 21, 2013
Introduction

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Now we shall study what happens when we reformulate the problem in a reverse fashion. We shall determine when a state variable reaches a predetermined value, or more precisely:

Given that the state variable $x_i$ currently assumes the value $x_i(t_k)$, we would like to determine the shortest time distance $h$, such that

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As we are simulating a continuous system on a digital computer, we need to discretize something, as no digital computer can compute infinitely many state changes within a finite time interval.

Until now, we always discretized the time axis, while keeping the state variables continuous. In the sequel, we shall discretize (quantize) the state variables, while keeping the time axis continuous.
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- The integration method will use a *variable step size*, and the step size in use will depend on the gradient of that state variable.

- The step size $h$ will be different for each state variable $x$.

- We shall no longer be able to represent the discretized system by a set of *difference equations*, and we shall *lose the linearity* of the discretized system when approximating a linear continuous system.

\[
\dot{x} = A \cdot x \quad \not\Rightarrow \quad x_{k+1} = F \cdot x_k
\]
Let us start by considering the following first-order system:

\[ \dot{x}_a(t) = -x_a(t) + 10 \cdot \varepsilon(t - 1.76) \]

with initial condition \( x_a(t_0 = 0) = 10. \)
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where \( q(t) \triangleq \text{floor}[x(t)] \) is the integer part of the variable \( x(t) > 0 \).
Space Discretization: A Simple Example

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The latter model can be simulated very easily.
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Simulation of Quantized System

$q(t), x(t)$

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We were able to complete the simulation in 17 very simple steps, thereby obtaining the exact solution of the quantized system.
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The solution of the quantized system is similar to that of the original continuous system.
Simulations of Quantized and Original Systems

$x(t)$

$x_a(t)$

Time

$x_a(t)$, $x(t)$
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DEVS stands for *Discrete EVent System specification*. The formalism was first introduced in the 1970s by *Bernard Zeigler*.

All systems, the input/output behavior of which can be described by *sequences of discrete events*, can be represented using the DEVS formalism.
The Definition of DEVS

Atomic DEVS Models

A DEVS model processes a sequence of input events and, in reaction to those events and its own initial discrete state, generates a sequence of output events.
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![Diagram of DEVS model](image)
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An **atomic DEVS model** is defined by the structure:

\[ M = (X, Y, S, \delta_{int}, \delta_{ext}, \lambda, ta) \]
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- **X** is the set of input values.
- **Y** is the set of output values.
- **S** is the set of state values.
- \( \delta_{int}(), \delta_{ext}(), \lambda(), \) and \( ta() \) are functions defining the dynamics of the system.
The Definition of DEVS II

The Behavior of an Atomic DEVS Model

\[
\begin{align*}
S & \quad s_2 &= \delta_{\text{int}}(s_1) \\
& \quad s_4 &= \delta_{\text{int}}(s_3) \\
& \quad s_3 &= \delta_{\text{ext}}(s_2, e, x_1) \\
Y & \quad y_1 &= \lambda(s_1) \\
& \quad y_2 &= \lambda(s_3)
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- $\delta_{\text{int}}(s)$ is the *internal transition function*.
- $\delta_{\text{ext}}(s, e, x)$ is the *external transition function*.
- $\lambda(s)$ is the *output function*.
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\( \delta_{int}(s) \) is the internal transition function.

\( \delta_{ext}(s, e, x) \) is the external transition function.

\( \lambda(s) \) is the output function.

\( ta(s) \) is the time advance function.
The Definition of DEVS III
The Specification of an Atomic DEVS Model

Each possible state $s$ ($s \in S$) has an associated *time advance* calculated by the *time advance function* $ta(s)$ ($ta(s) : S \to \mathbb{R}_0^+$). The time advance is a non-negative real number, determining how long the system remains in a given state in absence of input events.
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▶ If the state adopts the value \( s_1 \) at time \( t_1 \), after \( ta(s_1) \) units of time (i.e., at time \( t_1 + ta(s_1) \)), the system performs an *internal transition*, taking it to a new state \( s_2 \). The new state is calculated as \( s_2 = \delta_{int}(s_1) \). Function \( \delta_{int} \) (\( \delta_{int} : S \rightarrow S \)) is called the *internal transition function*. 

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- When the state changes its value from $s_1$ to $s_2$, an *output event* is produced with the value $y_1 = \lambda(s_1)$. Function $\lambda$ ($\lambda : S \rightarrow Y$) is called the *output function*. In this way, the functions $ta$, $\delta_{\text{int}}$ and $\lambda$ define the *autonomous behavior of a DEVS model*. 
When an input event arrives, the state changes instantaneously. The new state value depends not only on the value of the input event, but also on the previous state value and the elapsed time since the last transition. If the system assumes the state value $s_2$ at time $t_2$, and subsequently, an input event arrives at time $t_2 + e < ta(s_2)$ with value $x_1$, the new state is calculated as $s_3 = \delta_{ext}(s_2, e, x_1)$. In this case, we say that the system performs an external transition. Function $\delta_{ext}$ ($\delta_{ext} : S \times R_0^+ \times X \rightarrow S$) is called the external transition function. No output event is produced during an external transition.
Let us consider the following simple example: A system receives positive numbers in an asynchronous way. After it received a number $x$, it generates an output event with the number $x/2$ after $3 \cdot x$ time units.
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A DEVS model that correctly represents this behavior is the following:

$$M_F = (X, Y, S, \delta_{int}, \delta_{ext}, \lambda, ta), \text{ where}$$

- $X = Y = S = \mathbb{R}^+$
- $\delta_{int}(s) = \infty$
- $\delta_{ext}(s, e, x) = x$
- $\lambda(s) = s/2$
- $ta(s) = 3 \cdot s$
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Observe that the state can assume a time advance equal to $\infty$. When this occurs, we say that the system is in a passive state, since it will no longer change its state, unless and until it receives an input event.
Let us analyze what happens with the model $M_1$ when it receives an input event trajectory. Consider for instance that input events occur at times $t = 1$, $t = 3$, and $t = 10$ with the values 2, 1, and 5, respectively. Suppose that initially we have $t = 0$, $s = \infty$ and $e = 0$. 
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Then, the following behavior would be observed:

**time $t = 0$:**
- $s = \infty$
- $e = 0$
- $ta(s) = ta(\infty) = \infty$

**time $t = 1^-$:**
- $s = \infty$
- $e = 1$

**time $t = 1$:**
- $s = \delta_{ext}(s, e, x) = \delta_{ext}(\infty, 1, 2) = 2$

**time $t = 1^+$:**
- $s = 2$
- $e = 0$
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Then, the following behavior would be observed:

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  $e = 0$
  
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- **time $t = 1$**:
  
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- **time $t = 1^+$**:
  
  $s = 2$
  
  $e = 0$
  
  $ta(s) = ta(2) = 6$

- **time $t = 3^-$**:
  
  $s = 2$
  
  $e = 2$

- **time $t = 3$**:
  
  $s = \delta_{\text{ext}}(s, e, x) = \delta_{\text{ext}}(2, 2, 1) = 1$

- **time $t = 3^+$**:
  
  $s = 1$
  
  $e = 0$
  
  $ta(s) = ta(1) = 3$

- **time $t = 6$**:
  
  Output event with value $\lambda(s) = \lambda(1) = 0.5$
  
  $s = \delta_{\text{int}}(s) = \delta_{\text{int}}(1) = \infty$

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Coupled DEVS Models

Atomic DEVS models can be coupled to form more complex models. The most simple manner for defining the coupling between DEVS models is through the use of *input and output ports*. 
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![Coupled DEVS Models Diagram](image-url)
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etc.
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- etc.

The resulting coupled model $N$ can be used as if it were a new atomic model.
Example: DEVS Model of a Static Function

Let us consider a system that calculates a *static function* \( f(u_0, u_1) \), where \( u_0 \) and \( u_1 \) are real-valued piecewise constant trajectories generated by other subsystems. We can represent piecewise constant trajectories by *sequences of events*, if we relate each event to a change in the trajectory value.
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Using this idea, we can build the following atomic DEVS model:

$$MF = (X, Y, S, \delta_{int}, \delta_{ext}, \lambda, ta),$$

where

$$X = Y = \mathbb{R} \times \mathbb{N}_0$$

$$S = \mathbb{R}^2 \times \mathbb{R}_0^+$$

$$\delta_{int}(s) = \delta_{int}(u_0, u_1, \sigma) = (u_0, u_1, \infty)$$

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$$ta(s) = ta(u_0, u_1, \sigma) = \sigma$$

where:

$$\tilde{s} = \begin{cases} 
(x_v, u_1, 0) & \text{if } p = 0 \\
(u_0, x_v, 0) & \text{otherwise}
\end{cases}$$
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Example: DEVS Model of a Static Function

Let us consider a system that calculates a \textit{static function} \( f(u_0, u_1) \), where \( u_0 \) and \( u_1 \) are real-valued piecewise constant trajectories generated by other subsystems. We can represent piecewise constant trajectories by \textit{sequences of events}, if we relate each event to a change in the trajectory value.

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- When an input event arrives, it is assigned the value $\sigma = 0$. In this way, an immediate output event is being scheduled.
DEVS models can be simulated with a simple ad-hoc program written in any language. In fact, the simulation of a DEVS model is not much more complicated than that of a discrete-time model.
Simulation of DEVS Models

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3. Propagate the output event produced by $d^*$ to all atomic models connected to it through its output ports while executing the corresponding external transition functions. Then return to step 1 above.
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A routine called *DEVS-simulator* is associated with each atomic DEVS model, and a different routine called *DEVS-coordinator* is related to each coupled DEVS model. At the top of the hierarchy, there is a routine called *DEVS-root-coordinator* that manages the global simulation time.
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There exist several software tools that support directly the simulation of DEVS models. The one that we shall be using is called *PowerDEVS*. It was developed by *Ernesto Kofman* at the Universidad Nacional de Rosario (Argentina). It is the DEVS modeling and simulation environment that is most suitable for our purposes.
DEVS and Continuous System Simulation

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We can divide the quantized continuous system into:

a *dynamic system*:

\[
\dot{x}(t) = d_x(t) \\
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and a *static function*:

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d_x(t) = -q(t) + u(t)
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where:

$$\tilde{\sigma} = \begin{cases} \frac{q+1-x}{x_v} & \text{if } x_v > 0 \\ \frac{q-x}{x_v} & \text{if } x_v < 0 \\ \infty & \text{otherwise} \end{cases}$$
PowerDEVS Model of a Quantized System

The DEVS models $M_F$ (called *static function*) and $M_{QI}$ (called *quantized integrator*) are graphically represented as *PowerDEVS blocks*.
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We can model a generic time-invariant quantized system using DEVS models of the static function and quantized integrator types.
Quantized Systems: Illegitimacy

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We notice that \( q(t) \) oscillates between 10 and 9 with *infinite frequency*. For this reason, the DEVS model enters an *infinite loop*, and the simulation cannot advance.
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Luckily, this problem can be solved easily by adding *hysteresis*.
If we add *hysteresis* to the relationship between $x(t)$ and $q(t)$, the oscillations in $q(t)$ can only be produced by *large oscillations* in $x(t)$ that cannot occur instantaneously, as long as the magnitude of the state derivatives remains bounded.
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**Definition (Function of Quantization with Hysteresis)**

Given an ordered sequence of increasing real-valued numbers $(\ldots, Q_{-1}, Q_0, Q_1, \ldots)$, we say that $q(t)$ is related to $x(t)$ through a quantization function with hysteresis, if:

$$ q(t) = \begin{cases} 
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Quantization Functions with Hysteresis

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The discrete values $Q_k$ are called *quantization levels*, and the distance $Q_{k+1} - Q_k$ is called *quantum*. The quantum is often chosen constant. $\varepsilon_k$ is the *hysteresis width*. 
The graph depicted below shows a quantization function with hysteresis with a uniform quantum.

\[ q(t) \]

\[ Q_k \quad Q_{k+1} \]

\[ \varepsilon_k \]

\[ x(t) \]
QSS Method: Definition

Given the *time-invariant continuous system*:

\[
\begin{align*}
\dot{x}_1 &= f_1(x_1, x_2, \ldots, x_n, u_1, \ldots, u_m) \\
\vdots \\
\dot{x}_n &= f_n(x_1, x_2, \ldots, x_n, u_1, \ldots, u_m)
\end{align*}
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\end{align*}
\]

approximated by the \textit{quantized state system (QSS)}:

\[
\begin{align*}
\dot{x}_1 &= f_1(q_1, q_2, \cdots, q_n, u_1, \cdots, u_m) \\
& \quad \cdots \\
\dot{x}_n &= f_n(q_1, q_2, \cdots, q_n, u_1, \cdots, u_m)
\end{align*}
\]
QSS Method: Definition

Given the *time-invariant continuous system*:

\[
\begin{align*}
\dot{x}_a &= f_1(x_{a1}, x_{a2}, \cdots, x_{an}, u_1, \cdots, u_m) \\
&\vdots \\
\dot{x}_n &= f_n(x_{a1}, x_{a2}, \cdots, x_{an}, u_1, \cdots, u_m)
\end{align*}
\]

approximated by the *quantized state system (QSS)*:

\[
\begin{align*}
\dot{x}_1 &= f_1(q_1, q_2, \cdots, q_n, u_1, \cdots, u_m) \\
&\vdots \\
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where each \( q_i \) is related to \( x_i \) by a *hysteretic quantization function*. 
QSS Method: Definition

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\[
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\dot{x}_a_1 &= f_1(x_a_1, x_a_2, \ldots, x_a_n, u_1, \ldots, u_m) \\
\vdots \\
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\[
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\dot{q}_1 &= f_1(q_1, q_2, \ldots, q_n, u_1, \ldots, u_m) \\
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The QSS can be represented by the following block diagram:
QSS Method: Definition

Given the time-invariant continuous system:

\[ \dot{x}_{a_1} = f_1(x_{a_1}, x_{a_2}, \cdots, x_{a_n}, u_1, \cdots, u_m) \]
\[ \vdots \]
\[ \dot{x}_{a_n} = f_n(x_{a_1}, x_{a_2}, \cdots, x_{a_n}, u_1, \cdots, u_m) \]

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where each \( q_i \) is related to \( x_i \) by a hysteretic quantization function.

The QSS can be represented by the following block diagram:

As before, the QSS can be subdivided into static functions and quantized integrators.
DEVS Representation of a QSS

The DEVS models of the *static functions* are the same as before ($M_F$).
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DEVS Representation of a QSS

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The DEVS model of the *quantized integrators* changes a bit due to the presence of *hysteresis*:

\[
M_{HQI} = (X, Y, S, \delta_{int}, \delta_{ext}, \lambda, ta), \text{ where}
\]

\[
X = Y = \mathbb{R} \times \mathbb{N}; \quad S = \mathbb{R}^2 \times \mathbb{Z} \times \mathbb{R}^+_0
\]

\[
\delta_{int}(s) = \delta_{int}(x, d_x, k, \sigma) = (x + \sigma \cdot d_x, d_x, k + \text{sign}(d_x), \sigma_1)
\]

\[
\delta_{ext}(s, e, x_u) = \delta_{ext}(x, d_x, k, \sigma, e, x_v, p) = (x + e \cdot d_x, x_v, k, \sigma_2)
\]

\[
\lambda(s) = \lambda(x, d_x, k, \sigma) = (Q_{k+\text{sign}(d_x)}, 0)
\]

\[
ta(s) = ta(x, d_x, k, \sigma) = \sigma
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DEVS Representation of a QSS

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\[ ta(s) = ta(x, d_x, k, \sigma) = \sigma \]

with:

\[ \sigma_1 = \begin{cases} 
Q_{k+2} - \frac{u + \sigma \cdot d_x}{(x + \sigma \cdot d_x) - (Q_k - 1 - \varepsilon)} 
& \text{if } d_x > 0 \\
\infty 
& \text{if } d_x < 0 \\
\infty 
& \text{if } d_x = 0 
\end{cases} \]

\[ \sigma_2 = \begin{cases} 
Q_{k+1} - \frac{x + e \cdot d_x}{(x + e \cdot d_x) - (Q_k - \varepsilon)} 
& \text{if } x_v > 0 \\
\infty 
& \text{if } x_v < 0 \\
\infty 
& \text{if } x_v = 0 
\end{cases} \]
Simulation with QSS

In order to simulate a model using the QSS algorithm, we begin by choosing the quantum to be used by each state variable, i.e., by each hysteretic quantized integrator.
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However, PowerDEVS already comes with a library of pre-coded models of hysteretic quantized integrators (the user only needs to choose the quantum) and many different frequently used static functions (summers, limiters, etc.).
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It usually suffices to graphically construct the block diagram describing the system, choosing the quantum used by each of the state variables, and dragging the appropriate static functions from the graphical library and dropping them into the diagram window.
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It should be mentioned, however, that the QSS algorithm is independent of DEVS. We chose DEVS for the implementation of the QSS method, because DEVS simplified our work. However, we could have programmed the QSS method also independently of DEVS using any other event description formalism.
Simulation with QSS: An Illustrative Example

Let us consider the following second-order system and its QSS approximation:

\[
\begin{align*}
\dot{x}_1(t) &= x_2(t) & \dot{x}_1(t) &= q_2(t) \\
\dot{x}_2(t) &= 1 - x_1(t) - x_2(t) & \dot{x}_2(t) &= 1 - q_1(t) - q_2(t)
\end{align*}
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To simulate this system, we simply construct the block diagram using the hysteretic quantized integrator and the appropriate static functions of PowerDEVS:
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- The quantum and the hysteresis are parameters of each integrator (here: \(Q_{k+1} - Q_k = \Delta Q = \epsilon_k = 0.05\)).
- The QSS method intrinsically exploits sparsity (events are only propagated between directly connected blocks).
Simulation with QSS: An Illustrative Example II

The *simulation results* are shown below:
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![Results of the QSS Simulation](image)

- $x_1(t), q_1(t)$
- $x_2(t), q_2(t)$
The simulation results are shown below:

The trajectories of the state variables $x_i(t)$ are piecewise linear.
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The trajectories of the state variables $x_i(t)$ are piecewise linear.

The trajectories of the quantized states $q_i(t)$ are piecewise constant.

The presence of the hysteresis is easy to observe where the signs of the state derivatives $\dot{x}_i(t)$ change.

The obtained solution is quite close to the analytical solution.
In this presentation, we introduced a new type of discretization. Instead of *discretizing the time*, we proposed a *quantization of the state variables*.
Conclusions

▶ In this presentation, we introduced a new type of discretization. Instead of *discretizing the time*, we proposed a *quantization of the state variables*.

▶ We then outlined a new *numerical integration algorithm* based on this idea, the *QSS algorithm*, that operates on *quantized states with hysteresis*.
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We then outlined a new numerical integration algorithm based on this idea, the QSS algorithm, that operates on quantized states with hysteresis.

QSS simulations are intrinsically asynchronous. Each state variable changes its value independently of the other state variables.
Conclusions

- In this presentation, we introduced a new type of discretization. Instead of \textit{discretizing the time}, we proposed a \textit{quantization of the state variables}.

- We then outlined a new \textit{numerical integration algorithm} based on this idea, the \textit{QSS algorithm}, that operates on \textit{quantized states with hysteresis}.

- QSS simulations are \textit{intrinsically asynchronous}. Each state variable changes its value independently of the other state variables.

- The QSS algorithm exploits the \textit{sparsity of the model topology}. Events are propagated only between blocks that are directly connected.
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- QSS simulations are \textit{intrinsically asynchronous}. Each state variable changes its value independently of the other state variables.

- The QSS algorithm exploits the \textit{sparsity of the model topology}. Events are propagated only between blocks that are directly connected.

- Unfortunately, the QSS algorithm cannot be easily programmed as a \texttt{Matlab} function. Instead, we also introduced a new tool, \texttt{PowerDEVS}, that has been specifically designed for the numerical simulation of continuous systems using QSS algorithms.
